

# THE RATIONAL HOMOTOPY TYPE OF $(n-1)$ -CONNECTED $(4n-1)$ -MANIFOLDS

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**ABSTRACT.** We define the Bianchi-Massey tensor on the degree  $n$  cohomology with rational coefficients of a topological space  $X$  as a linear map  $\mathcal{B} \rightarrow H^{4n-1}(X)$ , where  $\mathcal{B}$  is a subspace of  $H^n(X)^{\otimes 4}$  determined by the cup product  $H^n(X) \times H^n(X) \rightarrow H^{2n}(X)$ . If  $M$  is a closed  $(n-1)$ -connected  $(4n-1)$ -manifold (and  $n \geq 2$ ) then its rational homotopy type is determined by its cohomology algebra and Bianchi-Massey tensor, and  $M$  is formal if and only if the Bianchi-Massey tensor vanishes.

We use the Bianchi-Massey tensor to show that there are many  $(n-1)$ -connected  $(4n-1)$ -manifolds that are not formal but have no non-zero Massey products, and to present a classification of simply-connected 7-manifolds up to finite ambiguity.

## 1. INTRODUCTION

This paper is concerned with the rational homotopy theory of closed  $(n-1)$ -connected  $(4n-1)$ -manifolds for  $n \geq 2$ . A continuous map  $f : X \rightarrow Y$  is a rational homotopy equivalence if the induced maps  $f_* : \pi_k(X) \otimes \mathbb{Q} \rightarrow \pi_k(Y) \otimes \mathbb{Q}$  are isomorphisms. If the spaces are simply-connected then this condition is equivalent to  $f^* : H^*(Y) \rightarrow H^*(X)$  being an isomorphism of the cohomology algebras (throughout the paper, we use cohomology with rational coefficients unless explicitly stated otherwise). A further fundamental rational homotopy invariant is the Massey product structure on  $H^*(X)$ . In particular, Massey products are an obstruction to  $X$  being *formal* in the sense of Sullivan [22] (see §3.2).

Miller [19] proved that, for  $n \geq 2$ , any closed  $(n-1)$ -connected manifold of dimension  $\leq 4n-2$  is formal. On the other hand, it was well known that there are examples of non-formal closed  $(n-1)$ -connected manifolds of dimension  $\geq 4n-1$  [18, 10]. A closed  $(n-1)$ -connected  $(4n-1)$ -manifold  $M$  therefore represents a borderline situation, the simplest non-trivial case from the point of view of rational homotopy. The only possible non-trivial Massey products in this case are triple products of elements in  $H^n(M)$ , taking values in (quotients of)  $H^{3n-1}(M)$ , but these do not in general suffice for determining the rational homotopy type of  $M$ .

The starting point of this paper is the definition, similar in style to Massey triple products, of what we term the *Bianchi-Massey tensor*. This captures precisely the information needed to determine the rational homotopy type of an  $(n-1)$ -connected  $(4n-1)$ -manifold, and in particular it is a *complete* obstruction to formality of such manifolds. Moreover, the Bianchi-Massey tensor can be computed directly from the cohomology ring of a coboundary  $W$  for  $M$  such that the restriction map  $H^n(W) \rightarrow H^n(M)$  is onto. This makes the determination of the rational homotopy type tractable for many examples.

**1.1. The Bianchi-Massey tensor.** We will define the Bianchi-Massey tensor on the degree  $n$  cohomology of a differential graded algebra. Let us first summarise some notation to manage the dependence of the symmetries of various spaces of tensors on  $\epsilon := (-1)^n$ , set up in further detail in §2.1. For a vector space  $V$  let  $\mathcal{G}_\epsilon^k V$  denote the quotient of  $V^{\otimes k}$  by relations of  $\epsilon$ -symmetry, *i.e.* the  $k$ th exterior power  $\Lambda^k V$  if  $\epsilon = -1$ , or the homogeneous degree  $k$  polynomials  $P^k V$  if  $\epsilon = +1$ . Let  $\mathcal{B}_\epsilon(V)$  denote the kernel of the full  $\epsilon$ -symmetrisation  $P^2 \mathcal{G}_\epsilon^2 V \rightarrow \mathcal{G}_\epsilon^4 V$ .

*Remark.*  $\mathcal{B}_\epsilon(V)$  can be identified with the subspace of  $V^{\otimes 4}$  consisting of tensors that satisfy the symmetries of the Riemann curvature tensor, in particular the first Bianchi identity, *cf.* Remark 2.3.

Given a graded algebra  $H^*$ , the product  $H^n \times H^n \rightarrow H^{2n}$  is  $\epsilon$ -commutative, so factors through a map  $c: \mathcal{G}_\epsilon^2 H^n \rightarrow H^{2n}$ . We let  $E_n := \ker c \subseteq \mathcal{G}_\epsilon^2 H^n$ , and

$$\mathcal{B}_n(H^*) := P^2 E_n \cap \mathcal{B}_\epsilon(H^n). \quad (1)$$

When  $H^*$  is the cohomology algebra of a topological space or differential graded algebra  $\bullet$ , we will use  $\mathcal{B}_n(\bullet)$  as a short-hand for  $\mathcal{B}_n(H^*(\bullet))$ .

Given a differential graded algebra  $(\mathcal{A}, d)$  over  $\mathbb{Q}$ , let  $\mathcal{Z}^k := \ker d \subseteq \mathcal{A}^k$ , the space of closed elements of degree  $k$ . Pick a right inverse  $\alpha: H^n(\mathcal{A}) \rightarrow \mathcal{Z}^n$  for the projection to cohomology. This induces a map  $\alpha^2: \mathcal{G}_\epsilon^2 H^n(\mathcal{A}) \rightarrow \mathcal{Z}^{2n}$ , taking exact values precisely on  $E_n$ ; there is a linear map  $\gamma: E_n \rightarrow \mathcal{A}^{2n-1}$  such that  $\alpha^2(e) = d\gamma(e)$  for  $e \in E_n$ . Now observe that the map  $P^2 E_n \rightarrow \mathcal{A}^{4n-1}$  induced by

$$E_n \otimes E_n \rightarrow \mathcal{A}^{4n-1}, e \otimes e' \mapsto \frac{1}{2}(\alpha^2(e)\gamma(e') + \gamma(e)\alpha^2(e'))$$

takes closed values on  $\mathcal{B}_n(\mathcal{A}) \subseteq P^2 E_n$ . It is easy to see that the induced map

$$\mathcal{F}: \mathcal{B}_n(\mathcal{A}) \rightarrow H^{4n-1}(\mathcal{A}) \quad (2)$$

is independent of the choice of  $\gamma$ . It is not as obvious, but nevertheless true, that  $\mathcal{F}$  is also independent of the choice of  $\alpha$  (Lemma 2.1).

**Definition 1.1.** The *Bianchi-Massey tensor* on the degree  $n$  cohomology of the DGA  $(\mathcal{A}, d)$  is the linear map (2).

If  $\phi: \mathcal{A} \rightarrow \mathcal{A}'$  is a DGA homomorphism then the induced map  $\phi_\#$  on cohomology maps  $E_n$  to  $E'_n$  and thus  $\mathcal{B}_n(\mathcal{A})$  to  $\mathcal{B}_n(\mathcal{A}')$ , and the Bianchi-Massey tensor is clearly functorial in the sense that the diagram below commutes:

$$\begin{array}{ccc} \mathcal{B}_n(\mathcal{A}) & \xrightarrow{\phi_\#} & \mathcal{B}_n(\mathcal{A}') \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F}' \\ H^{4n-1}(\mathcal{A}) & \xrightarrow{\phi_\#} & H^{4n-1}(\mathcal{A}') \end{array}$$

The definition of formality therefore immediately implies that the Bianchi-Massey tensors of  $\mathcal{A}$  must be trivial if  $\mathcal{A}$  is formal.

**1.2. Determining the rational homotopy type.** Any simply-connected topological space  $X$  has a rationalisation  $(X_\mathbb{Q}, f)$  (unique up to homotopy, see *e.g.* [11, Theorem 9.7]), which is a simply-connected space  $X_\mathbb{Q}$  together with a map  $f: X \rightarrow X_\mathbb{Q}$  such that  $f_*: \pi_k(X) \otimes \mathbb{Q} \rightarrow \pi_k(X_\mathbb{Q})$  are isomorphisms. Two spaces  $X$  and  $X'$  are rationally homotopy equivalent if and only if their rationalisations are homotopy equivalent.

For any topological space  $X$ , we can define the Bianchi-Massey tensor  $\mathcal{F}_X: \mathcal{B}_n(X) \rightarrow H^{4n-1}(X)$  in terms of the algebra  $\Omega_{\text{PL}}(X)$  of piecewise linear forms on  $X$  (see Sullivan [22, §7] or Félix–Halperin–Thomas [11, II 10(c)] for the definition of  $\Omega_{\text{PL}}(X)$ ). The following theorem identifies the Bianchi-Massey tensor as a complete obstruction to realising an isomorphism of the cohomology algebras of  $(n-1)$ -connected  $(4n-1)$ -manifolds by a rational homotopy equivalence. Such obstructions are studied more generally by Halperin and Stasheff [13].

**Theorem 1.2.** *For  $n \geq 2$ , the rational homotopy type of a closed  $(n-1)$ -connected  $(4n-1)$ -manifold  $M$  is determined by its cohomology algebra  $H^*(M)$  and the Bianchi-Massey tensor  $\mathcal{F}_M: \mathcal{B}_n(M) \rightarrow H^{4n-1}(M)$ . More precisely, if  $M$  and  $M'$  are closed  $(n-1)$ -connected  $(4n-1)$ -manifolds and  $G: H^*(M) \rightarrow H^*(M')$  is an isomorphism then there exists a homotopy equivalence  $g: M'_\mathbb{Q} \rightarrow M_\mathbb{Q}$  of the rationalisations such that  $G = g^*$  if and only if the diagram below commutes.*

$$\begin{array}{ccc} \mathcal{B}_n(M) & \xrightarrow{G} & \mathcal{B}_n(M') \\ \downarrow \mathcal{F}_M & & \downarrow \mathcal{F}_{M'} \\ H^{4n-1}(M) & \xrightarrow{G} & H^{4n-1}(M') \end{array}$$

We deduce Theorem 1.2 from Corollary 3.5, which characterises the minimal model of  $M$  in terms of  $H^*(M)$  and the Bianchi-Massey tensor. In Corollary 3.6, that picture of the minimal model also lets us understand when  $M$  is formal.

**Theorem 1.3.** *For  $n \geq 2$ , a closed  $(n-1)$ -connected  $(4n-1)$ -manifold  $M$  is formal if and only if its Bianchi-Massey tensor  $\mathcal{F}_M: \mathcal{B}_n(M) \rightarrow H^{4n-1}(M)$  is trivial.*

More generally, Kadeishvili [16] proved that one can define an  $A_\infty$ -algebra structure on the cohomology of any topological space, and by [23, Proposition 7] the space is formal if the choices in the definition can be made so that all the higher-order products vanish (a precise interpretation of the slogan that “a space is formal if and only if the Massey products vanish uniformly”.) Moreover, the equivalence class of the  $A_\infty$ -algebra structure determines the rational homotopy type.

One perspective on the Bianchi-Massey tensor is that it identifies the components of the  $A_\infty$ -structure that are significant in the context of  $(n-1)$ -connected  $(4n-1)$ -manifolds, see §2.4. Discarding the components that depend on choices is useful for understanding examples, and in the applications discussed below.

*Remark 1.4.* When  $n = 1$ , *i.e.* in the case of connected 3-manifolds, the Bianchi-Massey tensor is still a well-defined invariant, but for spaces that are not simply-connected the relation between the rational homotopy type and the minimal model is less straight-forward, and we will not consider it here.

**1.3. Realisation.** We next turn to the question of realisation of invariants of the above type. In §3.3 we apply a minor modification of Sullivan’s methods for the realisation of rational models by closed simply-connected manifolds [22, Theorem 13.2] to obtain the following result on realisation by  $(n-1)$ -connected manifolds.

**Theorem 1.5.** *Let  $H^*$  be a  $(4n-1)$ -dimensional rational Poincaré duality algebra with  $H^0 = \mathbb{Q}$  and  $H^k = 0$  for  $0 < k < n$ . Let  $p_* \in H^{4*}$ , and let  $\mathcal{F}: \mathcal{B}_n(H) \rightarrow H^{4n-1}$  be a linear map. Then there is a closed smooth  $(n-1)$ -connected  $(4n-1)$ -manifold  $M$  with rational Pontrjagin classes  $p_*(M)$  and Bianchi-Massey tensor  $\mathcal{F}_M$  such that*

$$(H^*(M), p_*(M), \mathcal{F}_M) \cong (H^*, p_*, \mathcal{F}).$$

We can also consider the problem of *integral* realisation of Bianchi-Massey tensors. For free abelian groups  $F^n$  and  $F^{2n}$  with a homomorphism  $\mathcal{G}_\epsilon^2 F^n \rightarrow F^{2n}$ , we can define  $\mathcal{B}_n(F)$  entirely analogously to (1). When  $F^*$  is the free part of  $H^*(M; \mathbb{Z})$ , we abbreviate this to  $\mathcal{B}_n(M; \mathbb{Z})$ . We henceforth implicitly assume that all manifolds are oriented and define the “integral restriction”  $\bar{\mathcal{F}}_M: \mathcal{B}_n(M; \mathbb{Z}) \rightarrow \mathbb{Q}$  as the composition  $\mathcal{B}_n(M; \mathbb{Z}) \rightarrow \mathcal{B}_n(M) \rightarrow H^{4n-1}(M) \rightarrow \mathbb{Q}$ , where the second map is  $\mathcal{F}_M$  and the third is integration over the fundamental class.

*Remark 1.6.* Treating the Bianchi-Massey tensor of a closed oriented  $(4n-1)$ -manifold  $M$  as an element of  $\mathcal{B}_n(M)^*$  is not natural in the context of Theorem 1.2—since the fundamental class is not invariant under rational homotopy equivalence—but it is in the context of diffeomorphism classification and/or cohomology with integer coefficients, where  $\mathcal{F}_M$  and  $\bar{\mathcal{F}}_M$  are equivalent.

Even under our simplifying connectivity assumptions, the question of the realisation of integral Poincaré duality rings by simply-connected manifolds is a hard problem, and we focus instead on realisation of the minimal integral data required to support the Bianchi-Massey tensor. Let  $TH^*(M; \mathbb{Z}) \subset H^*(M; \mathbb{Z})$  be the torsion subgroup,  $FH^*(M; \mathbb{Z}) := H^*(M; \mathbb{Z})/TH^*(M; \mathbb{Z})$  the free quotient and  $\tilde{\mathcal{C}}_M: \mathcal{G}_\epsilon^2 H^n(M; \mathbb{Z}) \rightarrow FH^{2n}(M; \mathbb{Z})$  the cup product modulo torsion.

**Theorem 1.7.** *Let  $F^n$  and  $F^{2n}$  be free abelian groups. For any homomorphisms  $\bar{\mathcal{C}}: \mathcal{G}_\epsilon^2 F^n \rightarrow F^{2n}$  and  $\bar{\mathcal{F}}: \mathcal{B}_n(F) \rightarrow \mathbb{Q}$ , there exists some closed  $(n-1)$ -connected  $M^{4n-1}$  with isomorphisms  $H^n(M; \mathbb{Z}) \cong F^n$  and  $FH^{2n}(M; \mathbb{Z}) \cong F^{2n}$  which identify  $(\tilde{\mathcal{C}}_M, \bar{\mathcal{F}}_M) \cong (\bar{\mathcal{C}}, \bar{\mathcal{F}})$ . In addition, if  $\bar{\mathcal{F}}_M$  is integer-valued then we may take  $H^{2n}(M; \mathbb{Z}) \cong FH^{2n}(M; \mathbb{Z}) \cong F^{2n}$ .*

In §4 we first describe how the Bianchi-Massey tensor of the boundary of a compact  $4n$ -manifold  $W$  can be computed in terms of the cohomology ring of  $W$ , and then prove Theorem 1.7 by constructing the required  $M$  as a boundary.

**1.4. Classification up to finite ambiguity.** One of the motivations of Sullivan's work on rational homotopy theory is the principle that the rational homotopy type of a simply-connected manifold together with some characteristic class and integral data determines the diffeomorphism type up to finite ambiguity, *e.g.* [22, Theorem 13.1] classifies smooth manifolds up to finite ambiguity in terms of their rational homotopy type, rational Pontrjagin classes, bounds on torsion and certain integral lattice invariants.

Kreck and Triantafyllou [18] work with less than the full rational homotopy type and present stronger results, *e.g.* [18, Theorem 2.2], where less of the lattice data is required explicitly, or can be replaced by parts of the integral cohomology ring. In the first instance, we are not too concerned about how little of the integral cohomology ring  $H^*(M; \mathbb{Z})$  one needs to remember; in Corollary 3.8 we explain how to deduce the following result, where  $\mathcal{B}_n(M; \mathbb{Z})^* := \text{Hom}(\mathcal{B}_n(M; \mathbb{Z}), \mathbb{Q})$  contains the Bianchi-Massey tensor  $\bar{\mathcal{F}}_M: \mathcal{B}_n(M; \mathbb{Z}) \rightarrow \mathbb{Q}$ .

**Theorem 1.8.** *Closed  $(n-1)$ -connected  $(4n-1)$ -manifolds  $M$  are classified up to finite ambiguity by their integral cohomology ring  $H^*(M; \mathbb{Z})$ , Pontrjagin classes  $p_k(M) \in H^{4k}(M; \mathbb{Z})$  and Bianchi-Massey tensor  $\bar{\mathcal{F}}_M \in \mathcal{B}_n(M; \mathbb{Z})^*$ ; i.e. given such an  $M$ , the set of  $(n-1)$ -connected  $(4n-1)$ -manifolds  $M'$  with a ring isomorphism  $G: H^*(M'; \mathbb{Z}) \rightarrow H^*(M; \mathbb{Z})$  such that  $G(p_k(M')) = p_k(M)$ , and  $G^\# \bar{\mathcal{F}}_M = \bar{\mathcal{F}}_{M'}$  contains only finitely many diffeomorphism classes.*

For simply-connected 7-manifolds, we can simplify the invariants from Theorem 1.8 to align them more closely with the realisation statement Theorem 1.7. In this case Poincaré duality means that  $H^*(M)$  is determined up to isomorphism by the rational cup square  $c: P^2 H^2(M) \rightarrow H^4(M)$ . Hence given a bound on the size of  $TH^*(M; \mathbb{Z})$ , the full cohomology ring  $H^*(M; \mathbb{Z})$  is determined up to finite ambiguity by the cup product modulo torsion,  $\tilde{c}: P^2 H^2(M; \mathbb{Z}) \rightarrow FH^4(M; \mathbb{Z})$ . Similarly  $p_1(M) \in H^4(M; \mathbb{Z})$  is determined up to finite ambiguity by its image  $\tilde{p}_1(M) \in FH^4(M; \mathbb{Z})$ , and by Theorem 1.8 we have

**Corollary 1.9.** *For all  $N \geq 0$ , closed 1-connected 7-manifolds  $M$  with  $|TH^*(M; \mathbb{Z})| < N$  are classified up to finite ambiguity by the cup product modulo torsion  $\tilde{c}_M: P^2 H^2(M; \mathbb{Z}) \rightarrow FH^4(M; \mathbb{Z})$ ,  $\tilde{p}_1(M) \in FH^4(M; \mathbb{Z})$  and Bianchi-Massey tensor  $\bar{\mathcal{F}}_M \in \mathcal{B}_n(M; \mathbb{Z})^*$ .*

In §5.3 we build on Theorem 1.7 to also study which  $\tilde{p}_1$  are realised. Proposition 5.6 gives a satisfactory understanding of which integral invariants are realised by simply-connected spin 7-manifolds, and we then discuss directions for a complete classification of such manifolds.

**1.5. Non-formal manifolds with only trivial Massey products.** We will elaborate on the relationship between the Bianchi-Massey tensor and the Massey triple products of elements of  $H^n(M)$  in §2.3. In summary, the triple product  $\langle x, z, y \rangle$  of  $x, y, z \in H^n(M)$  is defined if and only if  $xz = yz = 0 \in H^{2n}(M)$ , in which case  $\langle x, z, y \rangle$  is an element of the quotient  $H^{3n-1}(M)/(xH^{2n-1}(M) + yH^{2n-1}(M))$ . Therefore if  $w \in H^n(M)$  has  $xw = yw = 0$  then we get a well-defined

$$\langle x, z, y \rangle w \in \mathbb{Q}. \quad (3)$$

If the cup product  $c$  is trivial then (3) defines an element  $m \in (H^n(M)^*)^{\otimes 4}$ . It is  $\mathfrak{A}$ -symmetric under swapping  $x \leftrightarrow y$  or  $z \leftrightarrow w$ , symmetric under swapping both  $x \leftrightarrow z$  and  $y \leftrightarrow w$ , and also satisfies the Bianchi identity. The space of such tensors is naturally dual to  $\mathcal{B}_c(H^n(M))$ . When  $c$  is trivial,  $\mathcal{F}$  and  $m$  correspond under this duality. In general the duality can be used to determine all defined values of (3) from  $\mathcal{F}$ . In turn, we can use Poincaré duality to determine all Massey triple products from (3).

A basic point of this paper is that the  $\mathcal{F}$  side of this duality is more useful when  $c$  is non-trivial: then it is still the case that the Bianchi-Massey tensor determines the Massey triple products, but the converse is not true in general. In §5.2 we study how the spaces  $\mathcal{B}_n(M)$  of possible Bianchi-Massey tensors, and the set of defined Massey triple products, depend on the kernel  $E_n \subseteq \mathcal{G}_c^2 H^n(M)$  of  $c$ . Combined with the above realisation results, one finds that there are many examples of closed  $(4n-1)$ -manifolds with non-trivial Bianchi-Massey tensor but only trivial Massey products. The examples that are simplest to describe have  $n = 2k + 1$  odd.

*Example 1.10.* Combining Example 5.4 and Theorem 1.7, there exists for each  $k \geq 1$  a non-formal  $2k$ -connected  $(8k+3)$ -manifold  $M$  with  $H^{2k+1}(M; \mathbb{Z}) \cong \mathbb{Z}^4$  and  $H^{4k+2}(M; \mathbb{Z}) \cong \mathbb{Z}^3$  such that  $c : \Lambda^2 H^{2k+1}(M; \mathbb{Z}) \rightarrow H^{4k+2}(M; \mathbb{Z})$  is equivalent to

$$(f^1 \wedge f^2 - f^3 \wedge f^4, f^1 \wedge f^3 + f^2 \wedge f^4, f^1 \wedge f^4 - f^2 \wedge f^3)$$

(where  $f^i$  is a basis for  $H^{2k+1}(M; \mathbb{Z})^*$ ) even though all Massey products of a space with that cohomology ring are trivial—indeed, if  $x, y \in H^{2k+1}(M)$  have  $x \cup y = 0$  then  $x$  and  $y$  are linearly dependent.

*Example 1.11.* Combining Example 5.3 and Theorem 1.7, there exists for  $k \geq 1$  a non-formal  $(2k-1)$ -connected  $(8k-1)$ -manifold  $M$  with  $H^{2k}(M; \mathbb{Z}) \cong \mathbb{Z}^5$  and  $H^{4k}(M; \mathbb{Z}) \cong \mathbb{Z}^3$  such that  $c : P^2 H^{2k}(M; \mathbb{Z}) \rightarrow H^{4k}(M; \mathbb{Z})$  is equivalent to

$$2(x_1 x_4 + x_3 x_5, x_2 x_5 + x_3 x_4, x_1^2 + x_1 x_2 + x_2^2 + x_3^2 + x_4^2 + x_5^2),$$

(describing a homomorphism  $P^2 \mathbb{Z}^r \rightarrow \mathbb{Z}$  as a homogeneous quadratic polynomial with even cross terms) even though all Massey products of a space with that cohomology ring are trivial.

**1.6. Intrinsic formality.** A space  $X$  is said to be *intrinsically formal* if any space with cohomology algebra isomorphic to  $H^*(X)$  is rationally homotopy equivalent to  $X$ . In this case the only defined triple Massey products are ones of the form  $\langle x, y, x \rangle$ , which are trivial a priori. It is immediate from Theorem 1.3 that any  $(n-1)$ -connected  $(4n-1)$ -manifold  $M$  whose cohomology algebra has  $\mathcal{B}_n(M) = 0$  is intrinsically formal, while if  $\mathcal{B}_n(M) \neq 0$  then Theorem 1.5 lets us realise  $H^*(M)$  as the cohomology algebra of some non-formal space.

**Corollary 1.12.** *A closed  $(n-1)$ -connected  $(4n-1)$ -manifold  $M$  is intrinsically formal if and only if  $\mathcal{B}_n(M) = 0$ .*

Cavalcanti [6, Theorem 4] showed that if  $M$  is a closed  $(n-1)$ -connected  $(4n-1)$ -manifold and there is an element  $\varphi \in H^{2n-1}(M)$  such that  $H^n(M) \rightarrow H^{3n-1}(M)$ ,  $x \mapsto \varphi x$  is an isomorphism (“ $M$  has a hard Lefschetz property”) then  $M$  is formal if  $b_n(M) \leq 2$  and its Massey products vanish uniformly if  $b_n(M) \leq 3$ . As an illustration of our results we can make the following improvement.

**Theorem 1.13.** *Let  $M$  be a closed  $(n-1)$ -connected  $(4n-1)$ -manifold. If  $b_n(M) \leq 3$  and there is a  $\varphi \in H^{2n-1}(M)$  such that  $H^n(M) \rightarrow H^{3n-1}(M)$ ,  $x \mapsto \varphi x$  is an isomorphism then  $M$  is intrinsically formal.*

The algebraic claim that this relies on, Proposition 5.1, is essentially the same as the claim that the Ricci curvature of a manifold of dimension  $\leq 3$  determines the full Riemann curvature tensor.

**Acknowledgements.** The authors thank Manuel Amann, Marisa Fernández, Matthias Kreck and Vicente Muñoz for valuable discussions. DC acknowledges the support of the Leibniz Prize of Wolfgang Lück, granted by the Deutsche Forschungsgemeinschaft.

## 2. MASSEY PRODUCTS

We begin by proving that the Bianchi-Massey tensor is well-defined, as claimed in the introduction. We then set up some related linear algebra, and discuss the relation of the Bianchi-Massey tensor to ordinary Massey triple products. That will not be directly relevant until §4 and §5, but it provides some context for the Bianchi-Massey tensor.

**2.1. Well-definedness of the Bianchi-Massey tensor.** Let us first set up some notation that we will use throughout the paper. When  $n$  is implicit from the context, we set

$$\epsilon := (-1)^n, \quad \text{and} \quad \vartheta := (-1)^{n+1} \tag{4}$$

(the latter purely to avoid wide subscripts). For a vector space or  $\mathbb{Z}$ -module  $V$  we let  $\mathcal{G}_\epsilon^k V$  denote the quotient of  $V^{\otimes k}$  by relations of  $\epsilon$ -symmetry,  $P^k V := \mathcal{G}_+^k V$  and  $\Lambda^k V := \mathcal{G}_-^k V$ . Let  $\mathcal{B}_\epsilon(V)$  denote the kernel of the full  $\epsilon$ -symmetrisation  $P^2 \mathcal{G}_\epsilon^2 V \rightarrow \mathcal{G}_\epsilon^4 V$ .

For a submodule  $E \subseteq \mathcal{G}_\epsilon^2 V$ , let  $\mathcal{B}_\epsilon[E] := P^2 E \cap \mathcal{B}_\epsilon(V)$ . For a graded ring  $H^*$ , let  $\mathcal{B}_n(H^*) := \mathcal{B}_\epsilon[E_n]$  for  $E_n$  the kernel of the product map  $c : \mathcal{G}_\epsilon^2 H^n \rightarrow H^{2n}$  (this recovers (1)). As in the introduction, we abbreviate  $\mathcal{B}_n(H^*(\bullet))$  to  $\mathcal{B}_n(\bullet)$ , where  $\bullet$  is a DGA or a topological space.

In the proof that the Bianchi-Massey tensor is well-defined, we will use the following notation. Given linear maps  $\rho : V \rightarrow \mathcal{A}^r$  and  $\sigma : V \rightarrow \mathcal{A}^s$ , the bilinear maps

$$\begin{aligned} V \times V &\rightarrow \mathcal{A}^{r+s}, (v, w) \mapsto \frac{1}{2}(\rho(v)\sigma(w) + \rho(w)\sigma(v)), \\ V \times V &\rightarrow \mathcal{A}^{r+s}, (v, w) \mapsto \frac{1}{2}(\rho(v)\sigma(w) + \epsilon\rho(w)\sigma(v)) \end{aligned} \quad (5)$$

are symmetric and  $\epsilon$ -symmetric respectively, and we let  $\rho \cdot \sigma : P^2V \rightarrow \mathcal{A}^{r+s}$  and  $\rho\sigma : \mathcal{G}_\epsilon^2V \rightarrow \mathcal{A}^{r+s}$  denote the respective induced maps (and  $\rho^2 = \rho\rho$ , which can be non-zero when  $r = n \bmod 2$ ). Note that the usual Leibniz rules hold:  $d(\rho \cdot \sigma) = d\rho \cdot \sigma + (-1)^r \rho \cdot d\sigma : P^2V \rightarrow \mathcal{A}^{r+s+1}$  etc.

**Lemma 2.1.** *Let  $(\mathcal{A}, d)$  be a DGA, and  $\mathcal{Z} := \ker d$ . Fix  $n$ , and let  $E_n$  be the kernel of the product  $\mathcal{G}_\epsilon^2 H^n(\mathcal{A}) \rightarrow H^{2n}(\mathcal{A})$ . Choose a right inverse  $\alpha : H^n(\mathcal{A}) \rightarrow \mathcal{Z}^n$  for the projection to cohomology, and a linear map  $\gamma : E_n \rightarrow \mathcal{A}^{2n-1}$  such that  $d\gamma = \alpha^2$  on  $E_n$ . Then the linear map*

$$\gamma \cdot \alpha^2 : P^2E_n \rightarrow \mathcal{A}^{4n-1}$$

*takes values in  $\mathcal{Z}^{4n-1}$  on  $\mathcal{B}_n(\mathcal{A}) = \mathcal{B}_\epsilon[E_n]$ , and the induced map*

$$\mathcal{F} : \mathcal{B}_n(\mathcal{A}) \rightarrow H^{4n-1}(\mathcal{A})$$

*is independent of the choice of  $\gamma$  and  $\alpha$ . Thus Definition 1.1 of the Bianchi-Massey tensor is well-defined.*

*Proof.* Note that  $d(\gamma \cdot \alpha^2) : P^2E_n \rightarrow \mathcal{A}^{4n}$  is the restriction of  $\alpha^2 \cdot \alpha^2 : P^2\mathcal{G}_\epsilon^2 H^n(\mathcal{A}) \rightarrow \mathcal{A}^{4n}$ , which factors through  $\mathcal{G}_\epsilon^4 H^n(\mathcal{A})$ . Therefore  $d(\gamma \cdot \alpha^2)$  vanishes on the intersection of  $P^2E_n$  with the kernel of the  $\epsilon$ -symmetrisation  $P^2\mathcal{G}_\epsilon^2 H^n(\mathcal{A}) \rightarrow \mathcal{G}_\epsilon^4 H^n(\mathcal{A})$ , which is  $\mathcal{B}_n(\mathcal{A})$  by definition. Hence  $\gamma \cdot \alpha^2$  maps  $\mathcal{B}_n(\mathcal{A}) \rightarrow \mathcal{Z}^{4n-1}$  as claimed.

Now consider replacing  $\gamma$  by  $\gamma_\bullet = \gamma + \eta$ , for some  $\eta : E_n \rightarrow \mathcal{Z}^{2n-1}$ . Then  $\gamma_\bullet \cdot \alpha^2 - \gamma \cdot \alpha^2 = \eta \cdot \alpha^2 = -d(\eta \cdot \gamma)$  takes exact values on all of  $P^2E_n$ , so certainly  $\mathcal{F}$  is independent of the choice of  $\gamma$ , given the choice of  $\alpha$ .

Now consider replacing  $\alpha$  by  $\alpha_\bullet := \alpha + d\beta$  for some arbitrary linear map  $\beta : H^n(\mathcal{A}) \rightarrow \mathcal{A}^{n-1}$ . Then

$$\alpha_\bullet^2 - \alpha^2 = (d\beta)(2\alpha + d\beta)$$

on  $\mathcal{G}_\epsilon^2 H^n(M)$ , so a possible choice of  $\gamma_\bullet$  such that  $d\gamma_\bullet = \alpha_\bullet^2$  is

$$\gamma_\bullet := \gamma + \beta(2\alpha + d\beta).$$

The following equation is unambiguous if we insist that the concatenation products take priority over the  $\cdot$  products (so that the factors in the  $\cdot$  products always have degree  $2n-1$  or  $2n$ ). On  $P^2E_n$  we have

$$\begin{aligned} &\gamma_\bullet \cdot \alpha_\bullet^2 - \gamma \cdot \alpha^2 + d(\gamma \cdot \beta(2\alpha + d\beta) + \frac{2}{3}\beta\alpha \cdot \beta d\beta) \\ &= \beta(2\alpha + d\beta) \cdot \alpha^2 + \gamma \cdot (d\beta)(2\alpha + d\beta) + \beta(2\alpha + d\beta) \cdot (d\beta)(2\alpha + d\beta) \\ &\quad + d\gamma \cdot \beta(2\alpha + d\beta) - \gamma \cdot (d\beta)(2\alpha + d\beta) + \frac{2}{3}(d\beta)\alpha \cdot \beta d\beta - \frac{2}{3}\beta\alpha \cdot (d\beta)^2 \\ &= 2\alpha^2 \cdot \beta(2\alpha + d\beta) + \beta(2\alpha + d\beta) \cdot (d\beta)(2\alpha + d\beta) + \frac{2}{3}\beta d\beta \cdot (d\beta)\alpha - \frac{2}{3}\beta\alpha \cdot (d\beta)^2 \\ &= 4\alpha^2 \cdot \beta\alpha + 2\alpha^2 \cdot \beta d\beta + 4\beta\alpha \cdot (d\beta)\alpha + \frac{8}{3}\beta d\beta \cdot (d\beta)\alpha + \frac{4}{3}\beta\alpha \cdot (d\beta)^2 + \beta d\beta \cdot (d\beta)^2. \end{aligned}$$

The right hand side factors through a map  $\mathcal{G}_\epsilon^4 H^n(\mathcal{A}) \rightarrow \mathcal{A}^{4n-1}$ : the first term, the sum of the second and third terms, the sum of the fourth and fifth terms, and the sixth term are each fully  $\epsilon$ -symmetric. Hence the restriction of  $\gamma_\bullet \cdot \alpha_\bullet^2 - \gamma \cdot \alpha^2$  to  $\mathcal{B}_n(\mathcal{A})$  takes exact values, so  $\mathcal{F}$  is independent of the choice of  $\alpha$  too.  $\square$

*Remark 2.2.* An alternative way to describe the Bianchi-Massey tensor is to argue that the (unsymmetrised) map  $\gamma\alpha^2 : E_n \otimes E_n, e \otimes e' \mapsto \gamma(e)\alpha^2(e')$  induces a well-defined map  $Z \rightarrow H^{4n-1}(\mathcal{A})$ , where  $Z$  is the kernel of the full  $\epsilon$ -symmetrisation  $E_n \otimes E_n \rightarrow \mathcal{G}_\epsilon^4 H^n(\mathcal{A})$ . But  $\text{Alt}^2 E_n \subseteq Z$ , and the restriction of  $\gamma\alpha^2$  to  $\text{Alt}^2 E_n$  takes exact values. Thus the induced map descends to  $Z/\text{Alt}^2 E_n$ , which is naturally isomorphic to  $\mathcal{B}_n(\mathcal{A})$ .

**2.2. Some quadrilinear algebra.** Let  $V$  be finite dimensional vector space over  $\mathbb{Q}$ , and  $\epsilon = \pm 1$ . It is sometimes useful to consider the duals to the spaces  $P^2\mathcal{G}_\epsilon^2V$ ,  $\mathcal{B}_\epsilon(V)$  etc. Let  $\text{Grad}_\epsilon^k V^* \subseteq (V^*)^{\otimes k}$  denote the subspace of  $\epsilon$ -symmetric tensors,  $\text{Sym}^k \equiv \text{Grad}_+^k$  and  $\text{Alt}^k \equiv \text{Grad}_-^k$ . There is a natural duality between  $\text{Grad}_\epsilon^k V^*$  and  $\mathcal{G}_\epsilon^k V$ .

We may consider  $\text{Sym}^2 \text{Grad}_\epsilon^2 V^*$  as a subspace of  $(V^*)^{\otimes 4}$ , consisting of quadrilinear functions  $\alpha(x, y, z, w)$  that are  $\epsilon$ -symmetric under swapping  $x \leftrightarrow y$  or  $z \leftrightarrow w$ , and under swapping both  $x \leftrightarrow z$  and  $y \leftrightarrow w$ .

The adjoint to the natural projection  $P^2\mathcal{G}_\epsilon^2V \rightarrow \mathcal{G}_\epsilon^4V$  is the inclusion  $\text{Grad}_\epsilon^4 V^* \rightarrow \text{Sym}^2 \text{Grad}_\epsilon^2 V^*$  of fully  $\epsilon$ -symmetric 4-tensors, while the full  $\epsilon$ -symmetrisation  $\text{Sym}^2 \text{Grad}_\epsilon^2 V^* \rightarrow \text{Grad}_\epsilon^4 V^*$  is dual to

$$\mathcal{G}_\epsilon^4 V \rightarrow P^2\mathcal{G}_\epsilon^2 V, \quad xyzw \mapsto (xy)(zw) + (xz)(yw) + (xw)(yz)$$

(using concatenation to denote products in  $\mathcal{G}_\epsilon^k V$ ).

We let  $\check{\mathcal{B}}_\epsilon(V^*)$  denote the kernel of the full  $\epsilon$ -symmetrisation map  $\text{Sym}^2 \text{Grad}_\epsilon^2 V^* \rightarrow \text{Grad}_\epsilon^4 V^*$ . Equivalently,  $\alpha \in \text{Sym}^2 \text{Grad}_\epsilon^2 V^*$  belongs to  $\check{\mathcal{B}}_\epsilon(V^*)$  if and only if it satisfies the Bianchi identity

$$\alpha(x, y, z, w) + \alpha(y, z, x, w) + \alpha(z, x, y, w) = 0. \quad (6)$$

In particular,  $\check{\mathcal{B}}_-(V^*)$  is the space of tensors with the well-known symmetries of the Riemann curvature tensor.

*Remark 2.3.* The projections  $\text{Grad}_\epsilon^k V \rightarrow \mathcal{G}_\epsilon^k V$  and  $\check{\mathcal{B}}_\epsilon(V) \rightarrow \mathcal{B}_\epsilon(V)$  are isomorphisms, but that does not remain true if we replace  $V$  by a vector space over a field of arbitrary characteristic or by an abelian group.

Consider

$$\begin{aligned} \phi : (V^*)^{\otimes 4} &\rightarrow (V^*)^{\otimes 4} \\ (\phi\alpha)(x, y, z, w) &:= \alpha(x, z, y, w) - \alpha(z, y, x, w). \end{aligned}$$

$\phi$  maps  $\text{Sym}^2 \text{Sym}^2 V^*$  to  $\text{Sym}^2 \text{Alt}^2 V^*$  and vice versa, and it is easy to see that the images are contained in  $\check{\mathcal{B}}_-(V^*)$  and  $\check{\mathcal{B}}_+(V^*)$  respectively. Since

$$(\phi^2\alpha)(x, y, z, w) = 2\alpha(x, y, z, w) - \alpha(z, x, y, w) - \alpha(y, z, x, w),$$

we find that  $\frac{1}{3}\phi^2$  is a projection to the subspace of  $(V^*)^{\otimes 4}$  satisfying the Bianchi identity (6). That implies that the images of  $\phi : \text{Sym}^2 \text{Sym}^2 V^* \rightarrow \text{Sym}^2 \text{Alt}^2 V^*$  and  $\phi : \text{Sym}^2 \text{Alt}^2 V^* \rightarrow \text{Sym}^2 \text{Sym}^2 V^*$  are precisely  $\check{\mathcal{B}}_-(V^*)$  and  $\check{\mathcal{B}}_+(V^*)$ ; indeed,  $\phi : \check{\mathcal{B}}_+(V^*) \rightarrow \check{\mathcal{B}}_-(V^*)$  is an isomorphism, with inverse  $\frac{1}{3}\phi$ .

We obtain pairs of naturally dual exact sequences:

$$\begin{aligned} 0 &\longrightarrow \text{Grad}_\epsilon^4 V^* \longrightarrow \text{Sym}^2 \text{Grad}_\epsilon^2 V^* \xrightarrow{\phi} \text{Sym}^2 \text{Grad}_\epsilon^2 V^* \longrightarrow \text{Grad}_\epsilon^4 V^* \longrightarrow 0 \quad (7) \\ 0 &\longleftarrow \mathcal{G}_\epsilon^4 V \longleftarrow P^2\mathcal{G}_\epsilon^2 V \xleftarrow{\phi^\vee} P^2\mathcal{G}_\epsilon^2 V \longleftarrow \mathcal{G}_\epsilon^4 V \longleftarrow 0 \end{aligned}$$

*Remark 2.4.* This shows in particular that there is a natural perfect pairing  $\mathcal{B}_\epsilon(V) \times \check{\mathcal{B}}_\epsilon(V^*) \rightarrow \mathbb{Q}$ , and  $\mathcal{B}_\epsilon(V)$  and  $\check{\mathcal{B}}_\epsilon(V^*)$  can both be regarded as measuring a “symmetry defect” of elements of  $\text{Sym}^2 \text{Grad}_\epsilon^2 V^*$ .

*Remark 2.5.* If we take  $V$  to be a free  $\mathbb{Z}$ -module instead of a  $\mathbb{Q}$ -vector space, then these sequences remain exact for  $\epsilon = +1$ , but when  $\epsilon = -1$  there is some 3-torsion.

*Remark 2.6.* For  $\alpha, \beta \in \text{Grad}_\epsilon^2 V^*$ , we can define  $\alpha \circledast \beta \in \check{\mathcal{B}}_\epsilon(V^*)$  by

$$(\alpha \circledast \beta)(x, y, z, w) := \alpha(x, z)\beta(y, w) - \alpha(y, z)\beta(x, w) - \alpha(x, w)\beta(z, y) + \alpha(y, w)\beta(z, x). \quad (8)$$

Clearly the induced linear map  $\text{Sym}^2 \text{Grad}_\epsilon^2 V^* \rightarrow \check{\mathcal{B}}_\epsilon(V^*)$  equals the restriction of  $\phi$ . For  $\epsilon = +1$ , Besse [4, Definition 1.110] calls  $\circledast$  the *Kulkarni-Nomizu product*. Some of the results described here are therefore familiar in the context of Riemannian geometry.

**2.3. Massey triple products.** Let us recall the definition of Massey triple products. Let  $(\mathcal{A}, d)$  be a DGA, and  $\mathcal{Z} := \ker d$ . Suppose  $x_i \in H^{n_i}(\mathcal{A})$ , such that  $x_1x_2 = x_2x_3 = 0$ . Choose representatives  $\alpha_i \in \mathcal{Z}^{n_i}$  of the classes  $x_i$ . Then  $\alpha_1\alpha_2$  and  $\alpha_2\alpha_3$  are exact, say  $d\gamma_1$  and  $d\gamma_2$  respectively. Then  $\gamma_1\alpha_3 + (-1)^{n_1}\alpha_1\gamma_2 \in \mathcal{A}^{n_1+n_2+n_3-1}$  is closed, so represents a class in  $H^{n_1+n_2+n_3-1}(\mathcal{A})$ . The choices of  $\alpha_i$  and  $\gamma_j$  can change that class by elements of  $x_1H^{n_2+n_3-1}(\mathcal{A}) + x_3H^{n_1+n_2-1}(\mathcal{A})$ , but the image

$$\langle x_1, x_2, x_3 \rangle \in \frac{H^{n_1+n_2+n_3-1}(\mathcal{A})}{x_1H^{n_2+n_3-1}(\mathcal{A}) + x_3H^{n_1+n_2-1}(\mathcal{A})}$$

is well-defined, and that is the Massey triple product.

Now let us consider the case when  $n_1 = n_2 = n_3 = n$ . If  $x, y, z, w \in H^n(\mathcal{A})$  such that  $xz = yz = xw = yw = 0$ , then

$$\langle x, z, y \rangle w \in H^{4n-1}(\mathcal{A}) \quad (9)$$

is well-defined.

**Lemma 2.7** (cf. Hepworth [15, Lemma 3.1.4]).

- (i)  $\langle x, z, y \rangle w = \langle z, x, w \rangle y = \mathfrak{z}\langle y, z, x \rangle w = \mathfrak{z}\langle x, w, y \rangle z$
- (ii) If in addition  $xy = zw = 0$  then  $\langle x, z, y \rangle w + \langle y, x, z \rangle w = \langle z, y, x \rangle w = 0$

If the product  $H^n(\mathcal{A}) \times H^n(\mathcal{A}) \rightarrow H^{2n}(\mathcal{A})$  is trivial then setting  $m(x, y, z, w) := \langle x, z, y \rangle w$  defines a linear map  $H^n(\mathcal{A})^{\otimes 4} \rightarrow H^{4n-1}(\mathcal{A})$ , or equivalently an element of  $(H^n(\mathcal{A})^*)^{\otimes 4} \otimes H^{4n-1}(\mathcal{A})$ . Lemma 2.7(i) means that  $m$  is  $\mathfrak{z}$ -symmetric under swapping  $x \leftrightarrow y$  or  $z \leftrightarrow w$ , and also symmetric under swapping both  $x \leftrightarrow z$  and  $y \leftrightarrow w$ , so  $m \in \text{Sym}^2 \text{Grad}_2^2 H^n(\mathcal{A})^* \otimes H^{4n-1}(\mathcal{A})$ . Moreover, (ii) says that Bianchi identity holds, so  $m \in \check{\mathcal{B}}_3(H^2(\mathcal{A})^*) \otimes H^{4n-1}(\mathcal{A})$ .

The defined values of (9) can be recovered from the Bianchi-Massey tensor. Let  $xy$  denote the product in  $\mathcal{G}_\epsilon^2$ .

**Lemma 2.8.** If  $x, y, z, w \in H^n(\mathcal{A})$  such that  $xz = xw = yz = yw \in E_n$  (i.e. the products in  $H^{2n}(\mathcal{A})$  vanish) then

$$v := (xz)(yw) - (xw)(yz) \in \mathcal{B}_n(\mathcal{A}), \quad (10)$$

and

$$\mathcal{F}(v) = \langle x, z, y \rangle w. \quad (11)$$

**Definition 2.9.** We call elements of  $\mathcal{B}_n(\mathcal{A})$  of the form (10) *ordinary*.

If the product  $H^n(\mathcal{A}) \times H^n(\mathcal{A}) \rightarrow H^{2n}(\mathcal{A})$  is trivial, so that  $E_n = \mathcal{G}_\epsilon^2 H^n(\mathcal{A})$ , then (11) means that  $\mathcal{F}$  can be recovered from  $m$  using the duality  $\check{\mathcal{B}}_3(H^n(\mathcal{A})^*) \cong \mathcal{B}_\epsilon(H^n(\mathcal{A}))^*$ . Put differently, the surjectivity of the map  $\phi^\vee : P^2 \mathcal{G}_\epsilon^2 H^n(\mathcal{A}) \rightarrow \mathcal{B}_\epsilon(H^n(\mathcal{A}))$  implies that  $\mathcal{B}_\epsilon(H^n(\mathcal{A}))$  is generated by ordinary elements; if  $E_n = \mathcal{G}_\epsilon^2 H^n(\mathcal{A})$  then  $\mathcal{B}_n(\mathcal{A}) = \mathcal{B}_\epsilon(H^n(\mathcal{A}))$ , and (11) implies that  $\mathcal{F}$  can be recovered from the Massey triple products whenever  $\mathcal{B}_n(\mathcal{A})$  is generated by ordinary elements—but we will see in §5.2 that that is often *not* the case.

*Remark 2.10.* Suppose that  $\mathcal{A}$  is the DGA of piecewise linear forms on a closed oriented  $(4n-1)$ -manifold  $M$ , so that  $H^*(\mathcal{A}) \cong H^*(M)$  is a Poincaré duality algebra. Then  $H^{3n-1}(M)^* \cong H^n(M)$ , and for  $x, y \in H^n(M)$  the annihilator of  $xH^{2n-1}(M) + yH^{2n-1}(M) \subseteq H^{3n-1}(M)$  is precisely  $\{w \in H^{3n-1}(M) : xw = yw = 0\}$ . Hence for  $x, y, z \in H^n(M)$  such that  $xz = yz = 0$ , the triple product  $\langle x, z, y \rangle$  is completely determined by the values of  $\langle x, z, y \rangle w \in H^{4n-1}(M) \cong \mathbb{Q}$  for  $w$  such that  $xw = yw = 0$ , and hence by the Bianchi-Massey tensor.

**2.4. Relationship with  $A_\infty$ -structures.** For any DGA  $\mathcal{A}$ , one may define an  $A_\infty$ -structure on  $H^*(\mathcal{A})$ , which consists of a sequence of linear maps  $\mu_k : H^*(\mathcal{A})^{\otimes k} \rightarrow H^*(\mathcal{A})$  of degree  $2 - k$ , for  $k \geq 2$ , see e.g. Amann [2, §8.5] or Vallette [23].  $\mu_2$  is simply the product on the cohomology algebra, while the higher products are the ‘ $A_\infty$ -Massey products’. The definition of the higher products relies on arbitrary choices, but the structure is well-defined up to a suitable notion of  $A_\infty$ -isomorphism. There is also a notion of homotopy equivalence of  $A_\infty$  algebras, and two simply-connected spaces are rationally homotopy equivalent if and only if their cohomology rings are equivalent in that sense: see [23, Theorem 8] or [2, §8.5].



We shall only be concerned with  $\mu_3: H^n(\mathcal{A})^{\otimes 3} \rightarrow H^{3n-1}(\mathcal{A})$ , which may be defined as follows. Pick a right inverse  $\alpha: H^n(\mathcal{A}) \rightarrow \mathcal{Z}^n$  and a  $\gamma: \mathcal{Z}^{2n} \rightarrow \mathcal{A}^{2n-1}$  such that  $d\gamma: \mathcal{Z}^{2n} \rightarrow \mathcal{Z}^{2n}$  is a projection to the exact part, and set

$$\mu_3(x, z, y) := [\alpha(x)\gamma(\alpha(z)\alpha(y)) - \epsilon\gamma(\alpha(x)\alpha(z))\alpha(y)] \in H^{3n-1}(\mathcal{A}).$$

If  $xz = yz = 0$  then clearly

$$\mu_3(x, z, y) = \langle x, z, y \rangle \mod xH^n(\mathcal{A}) + yH^n(\mathcal{A})$$

(as the terminology suggests, such relations hold for the  $A_\infty$ -Massey products more generally). Now consider the 4-tensors  $\hat{\mu}_3 \in \text{Sym}^2 \text{Grad}_\epsilon^2 H^n(\mathcal{A})^* \otimes H^{4n-1}(\mathcal{A})$  and  $F \in \text{Sym}^2 \text{Grad}_\epsilon^2 H^n(\mathcal{A})^* \otimes H^{4n-1}(\mathcal{A})$  defined by

$$\hat{\mu}_3(x, z, y, w) := \mu_3(x, z, y)w - \mu_3(x, w, y)z \in H^{4n-1}(\mathcal{A}),$$

$$F(x, y, z, w) := [\gamma(\alpha(x)\alpha(y))\alpha(z)\alpha(w) + \alpha(x)\alpha(y)\gamma(\alpha(z)\alpha(w))] \in H^{4n-1}(\mathcal{A}).$$

Comparing with Definition 1.1, we see that the restriction of  $F$  to  $\mathcal{B}_n(\mathcal{A}) \subseteq \mathcal{B}_\epsilon(H^n(\mathcal{A})) \subseteq P^2 \mathcal{G}_\epsilon^2 H^n(\mathcal{A})$  equals the Bianchi-Massey tensor  $\mathcal{F}$ . On the other hand, in the notation of §2.2, we have  $\hat{\mu}_3 = \phi F$ , and hence  $\frac{1}{3}\phi\hat{\mu}_3$  is the projection of  $F$  to  $\check{\mathcal{B}}_\epsilon(H^n(\mathcal{A})^*)$ . In particular, the restrictions of  $\frac{1}{3}\hat{\mu}_3$  and  $F$  to  $\mathcal{B}_\epsilon(H^n(\mathcal{A}))$  agree. Thus the Bianchi-Massey tensor is completely determined by  $\mu_3$ .

Conversely,  $\mathcal{F}$  determines the components of  $\hat{\mu}_3$  that do not depend on choices. When  $\mathcal{A}$  is  $(n-1)$ -connected and  $(4n-1)$ -dimensional Poincaré, Theorem 1.2 tells us that  $\mathcal{F}$  in this way captures all the interesting data of the  $A_\infty$ -structure on  $H^*(\mathcal{A})$  that is needed to determine the homotopy type of  $\mathcal{A}$ .

### 3. THE RATIONAL HOMOTOPY TYPE

In this section we prove our main theorems on the significance of the Bianchi-Massey tensor for the rational homotopy type, and hence formality and diffeomorphism classification up to finite ambiguity, of  $(n-1)$ -connected  $(4n-1)$ -manifolds.

**3.1. Minimal Sullivan algebras.** We first classify  $(n-1)$ -connected  $(4n-1)$ -dimensional Poincaré minimal Sullivan algebras via their cohomology algebras and Bianchi-Massey tensors. Recall that a *minimal Sullivan algebra* is a DGA  $(\mathcal{M}, d)$  that is free as a graded algebra,  $\mathcal{M} \cong \Lambda V$ , and has a well-ordered basis  $\{v_\alpha\} \subset V$  such that  $dv_\alpha$  lies in the subalgebra generated by  $\{v_\beta : \beta < \alpha\}$ , and  $\alpha \leq \beta \Rightarrow \deg v_\alpha \leq \deg v_\beta$ . We are only interested in the case when  $\mathcal{M}$  is simply-connected. In this case, the minimality condition reduces to saying that  $\mathcal{M}$  is free, and

$$\text{for any } v \in \mathcal{M}^i, dv \text{ is a linear combination of products of elements of degree } < i. \quad (12)$$

**Definition 3.1.** Let  $H^*$  be a finite dimensional graded commutative algebra over the rationals.

- (i) We call  $\alpha \in (H^m)^*$  a *Poincaré class* if the linear map

$$\alpha \cap : H^i \rightarrow (H^{m-i})^*, \quad x \mapsto (y \mapsto \alpha(xy))$$

is an isomorphism for all  $i$ . We say that  $H^*$  is *m-dimensional Poincaré* if such an  $\alpha$  exists.

- (ii) For  $2k+1 \geq m$ , we call  $\alpha \in (H^m)^*$  a *k-partial Poincaré class* if  $\alpha \cap$  is an isomorphism for  $m-k \leq i \leq k$  and injective for  $i = k+1$  (and hence surjective for  $i = m-k+1$ ).

We call  $H^*$  *j-connected* if  $H^0 = \mathbb{Q}$  and  $H^k = 0$  for  $1 \leq k \leq j$ . A DGA  $\mathcal{A}$  is *m-dimensional Poincaré* if  $H^*(\mathcal{A})$  is *m-dimensional Poincaré* and is *j-connected* if  $\mathcal{A}^k = 0$  for  $1 \leq k \leq j$ .

The key to the role of the Bianchi-Massey tensor in this paper is the following existence and uniqueness result for minimal Sullivan algebras with prescribed Bianchi-Massey tensor.

#### Theorem 3.2.

- (i) For every  $(n-1)$ -connected  $(4n-1)$ -dimensional Poincaré algebra  $H^*$  and for every linear map  $\mathcal{F}: \mathcal{B}_n(H^*) \rightarrow H^{4n-1}$ , there exists an  $(n-1)$ -connected minimal Sullivan algebra  $\mathcal{M}$  with  $H^*(\mathcal{M}) = H^*$  and Bianchi-Massey tensor  $\mathcal{F}$ .

- (ii) If  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are  $(n-1)$ -connected  $(4n-1)$ -dimensional Poincaré minimal Sullivan algebras and  $G: H^*(\mathcal{M}_1) \rightarrow H^*(\mathcal{M}_2)$  is an isomorphism of the cohomology algebras then there is a DGA isomorphism  $\phi: \mathcal{M}_1 \rightarrow \mathcal{M}_2$  such that  $\phi_\# = G$  if and only if the diagram below commutes.

$$\begin{array}{ccc} \mathcal{B}_n(\mathcal{M}_1) & \xrightarrow{G} & \mathcal{B}_n(\mathcal{M}_2) \\ \downarrow \mathcal{F}_1 & & \downarrow \mathcal{F}_2 \\ H^{4n-1}(\mathcal{M}_1) & \xrightarrow{G} & H^{4n-1}(\mathcal{M}_2) \end{array}$$

Let us first outline the essence of the proof. The standard technique is to describe the degree  $i$  part  $V^i$  of the generating set of the minimal algebra  $\mathcal{M} = \Lambda V$  recursively in terms of the cohomology algebra and any further data (*i.e.* the Bianchi-Massey tensor in this case), using the relation (12). Let  $\mathcal{M}^{<i}$  denote the sub-DGA generated by elements of degree  $< i$ . Presenting  $V^i$  as the sum of its closed subspace  $C^i$  and a direct complement  $N^i$ , in the  $i$ th step of the recursion one identifies  $C^i$  and  $N^{i-1}$  with the kernel and cokernel of  $H^i(\mathcal{M}^{<i}) \rightarrow H^i(\mathcal{M})$ . In the setting of Theorem 3.2, this map is essentially determined by the cohomology algebra except for  $i = 3n - 1$  or  $4n - 1$ , where the Bianchi-Massey tensor appears. To prove the uniqueness statement (ii), one can argue that the generating set of any minimal algebra with the prescribed cohomology and Bianchi-Massey tensor can be described this way—while the description involves some arbitrary choices, those can be expressed in terms of the cohomology ring,

We will split the argument into two parts. First we explain that more generally, a minimal algebra  $\mathcal{M}$  that is  $m$ -dimensional Poincaré is essentially characterised by  $\mathcal{M}^{\leq k}$  for any  $2k \geq m - 1$ , together with the map  $H^m(\mathcal{M}^{\leq k}) \rightarrow H^m(\mathcal{M})$ . This is implicit in Kreck-Triantafyllou [18, Theorem 1.2]. Lemma 3.4 then shows that for  $(n-1)$ -connected  $(4n-1)$ -dimensional Poincaré minimal algebras, that data is characterised by the cohomology algebra and the Bianchi-Massey tensor.

**Proposition 3.3.** *Let  $k \geq 2$ ,  $m \leq 2k + 1$ .*

- (i) *Let  $\mathcal{M}$  be a minimal Sullivan algebra, and  $\alpha \in H^m(\mathcal{M})^*$  a Poincaré class. Then the image of  $\alpha$  in  $H^m(\mathcal{M}^{\leq k})^*$  is a  $k$ -partial Poincaré class.*
- (ii) *Let  $\mathcal{N}$  be a simply-connected minimal Sullivan algebra generated in degree  $\leq k$ , and let  $\alpha \in H^m(\mathcal{N})^*$  be a  $k$ -partial Poincaré class. Then there is a minimal Sullivan algebra  $\mathcal{M}$  with Poincaré class  $\alpha_{\mathcal{M}} \in H^m(\mathcal{M})^*$  and an isomorphism  $\tau: \mathcal{N} \rightarrow \mathcal{M}^{\leq k}$  such that  $\tau_\#^* \alpha_{\mathcal{M}} = \alpha$ .*
- (iii) *Let  $\mathcal{M}_1, \mathcal{M}_2$  be simply-connected minimal Sullivan algebras that are  $m$ -dimensional Poincaré. Let  $\tau: \mathcal{M}_1^{\leq k} \rightarrow \mathcal{M}_2^{\leq k}$  and  $G: H^m(\mathcal{M}_1) \rightarrow H^m(\mathcal{M}_2)$  be isomorphisms, such that the diagram below commutes.*

$$\begin{array}{ccc} H^m(\mathcal{M}_1^{\leq k}) & \xrightarrow{\tau_\#} & H^m(\mathcal{M}_2^{\leq k}) \\ \downarrow & & \downarrow \\ H^m(\mathcal{M}_1) & \xrightarrow{G} & H^m(\mathcal{M}_2) \end{array}$$

*Then there is an isomorphism  $\phi: \mathcal{M}_1 \rightarrow \mathcal{M}_2$  such that the restriction  $\phi^{\leq k}: \mathcal{M}_1^{\leq k} \rightarrow \mathcal{M}_2^{\leq k}$  equals  $\tau$  and  $\phi_\#: H^m(\mathcal{M}_1) \rightarrow H^m(\mathcal{M}_2)$  equals  $G$ .*

*Proof.* (i) Let  $\iota: \mathcal{M}^{\leq k} \hookrightarrow \mathcal{M}$  be the inclusion. Then the horizontal maps in the diagram

$$\begin{array}{ccc} H^i(\mathcal{M}^{\leq k}) & \xrightarrow{\iota_\#} & H^i(\mathcal{M}) \\ \downarrow \iota_\#^* \alpha \cap & & \downarrow \alpha \cap \\ H^{m-i}(\mathcal{M}^{\leq k})^* & \xleftarrow{\iota_\#^*} & H^{m-i}(\mathcal{M})^* \end{array}$$

are isomorphisms for  $m - k \leq i \leq k$  and injective for  $i = k + 1$ , *cf.* [18, p 359].

- (ii) If  $k \geq m$ , then constructing the generators  $V^i$  in degree  $i > k$  for  $\mathcal{M}$  is a trivial recursion:  $d$  maps  $V^i = N^i$  isomorphically to the closed subspace of the degree  $i + 1$  part of  $\mathcal{M}^{<i}$ . The

following claim, which lets us increase  $k$  inductively until we reach  $k = m$ , therefore proves the result.

There exists a minimal Sullivan algebra  $\mathcal{E}$  generated in degree  $\leq k+1$  with a  $(k+1)$ -partial Poincaré class  $\alpha_{\mathcal{E}} \in H^m(\mathcal{E})$  and an isomorphism  $\phi: \mathcal{N} \rightarrow \mathcal{E}^{\leq k}$  such that  $\phi_{\#}^* \alpha_{\mathcal{E}} = \alpha$ .

Let us first construct the generating space  $V$  for  $\mathcal{E}$ . We take the degree  $\leq k$  parts to equal those of  $\mathcal{N}$  (and define  $\phi$  to be the inclusion). Choose a direct complement  $C^{k+1}$  of the image of  $\alpha \cap: H^{k+1}(\mathcal{N}) \rightarrow H^{m-k-1}(\mathcal{N})^*$ , and set  $V^{k+1} := C^{k+1} \oplus N^{k+1}$ , where  $d: V^{k+1} \rightarrow \mathcal{N}^{k+2}$  has kernel  $C^{k+1}$  and maps  $N^{k+1}$  isomorphically to a subspace of  $\mathcal{Z}^{k+2}$  (the closed subspace of  $\mathcal{N}^{k+2}$ ) representing the kernel of  $\alpha \cap: H^{k+2}(\mathcal{N}) \rightarrow H^{m-k-2}(\mathcal{N})^*$ .

We need to study  $H^m(\mathcal{E})$ . Note that  $\mathcal{E}^m = \mathcal{N}^m \oplus V^{k+1} \otimes \mathcal{N}^{m-k-1}$ , and the closed subspace is contained in  $\mathcal{N}^m \oplus V^{k+1} \otimes \mathcal{Z}^{m-k-1}$ . Therefore  $H^m(\mathcal{E})$  can be written as a direct sum of the images of  $H^m(\mathcal{N})$  and  $C^{k+1} \otimes H^{m-k-1}(\mathcal{N})$  and a direct complement  $W$ .

Note that  $\mathcal{E}^{m-1} = \mathcal{N}^{m-1} \oplus V^{k+1} \otimes \mathcal{N}^{m-k-2}$ . Therefore we have  $d\mathcal{E}^{m-1} \cap C^{k+1} \otimes \mathcal{Z}^{m-k-1} = C^{k+1} \otimes d\mathcal{N}^{m-k-2}$ , so the map  $C^{k+1} \otimes H^{m-k-1}(\mathcal{N}) \rightarrow H^m(\mathcal{E})$  is injective. On the other hand, the kernel of  $H^m(\mathcal{N}) \rightarrow H^m(\mathcal{E})$  consists of classes represented by elements of  $\mathcal{N}^m \cap d(V^{k+1} \otimes \mathcal{N}^{m-k-2}) = dN^{k+1} \otimes \mathcal{Z}^{m-k-2}$ . Since this is contained in the kernel of  $\alpha$  by construction,  $\alpha$  factors through  $\phi_{\#}$ .

We can therefore define the restriction of  $\alpha_{\mathcal{E}}$  to the image of  $H^m(\mathcal{N})$  by requiring that  $\alpha = \phi_{\#}^* \alpha_{\mathcal{E}}$ . Further we define the restriction to  $C^{k+1} \otimes H^{m-k-1}(\mathcal{N})$  to be the natural map arising from  $C^{k+1}$  being a subspace of  $H^{m-k-1}(\mathcal{N})^*$ , and choose the restriction to  $W$  to be 0. It remains to prove that this  $\alpha_{\mathcal{E}} \in H^m(\mathcal{E})^*$  is a  $(k+1)$ -partial Poincaré class.

For  $m-k \leq i \leq k$ ,  $H^i(\mathcal{E}) \cong H^i(\mathcal{N})$ , and it is easy to see that  $\alpha_{\mathcal{E}} \cap$  is equivalent to the isomorphism  $\alpha \cap$ . Meanwhile  $H^{k+1}(\mathcal{E}) \cong H^{k+1}(\mathcal{N}) \oplus C^{k+1}$ , and  $\alpha_{\mathcal{E}} \cap: H^{k+1}(\mathcal{E}) \rightarrow H^{m-k-1}(\mathcal{E})^* \cong H^{m-k-1}(\mathcal{N})^*$  equals the injective map  $\alpha \cap$  on the  $H^{k+1}(\mathcal{N})$  summand, and the inclusion  $C^{k+1} \hookrightarrow H^{m-k-1}(\mathcal{N})^*$  on  $C^{k+1}$ . Since we chose  $C^{k+1}$  to be a direct complement of the image of  $\alpha \cap$  that means that  $\alpha_{\mathcal{E}} \cap$  is an isomorphism on  $H^{k+1}(\mathcal{E})$  too. Finally,  $H^{k+2}(\mathcal{E}) = \mathcal{Z}^{k+2}/dN^{k+1} \cong H^{k+2}(\mathcal{N})/\ker(\alpha \cap)$ , and  $\alpha_{\mathcal{E}} \cap: H^{k+2}(\mathcal{N})/\ker(\alpha \cap) \rightarrow H^{m-k-2}(\mathcal{E})^* \cong H^{m-k-2}(\mathcal{N})^*$  is the map induced by  $\alpha \cap$ , so injective.

(iii) follows by induction from the following claim.

Let  $\mathcal{E}_1$  and  $\mathcal{E}_2$  be minimal Sullivan algebras generated in degree  $\leq k+1$ , with  $(k+1)$ -partial Poincaré classes  $\alpha_j \in H^m(\mathcal{E}_j)$ . Suppose  $\tau: \mathcal{E}_1^{\leq k} \rightarrow \mathcal{E}_2^{\leq k}$  is an isomorphism such that  $\tau_{\#}^* \alpha_2 \in H^m(\mathcal{E}_1^{\leq k})$  equals the restriction of  $\alpha_1$ . Then there exists an isomorphism  $\phi: \mathcal{E}_1 \rightarrow \mathcal{E}_2$  such that  $\phi^{\leq k} = \tau$  and  $\phi_{\#}^* \alpha_2 = \alpha_1$ .

We can use the argument in (ii) to describe the generating spaces of  $\mathcal{E}_j$ . This involves choices of  $C_j^{k+1}$  and  $dN_j^{k+1}$ , and we choose  $C_2^{k+1} = \tau_{\#}(C_1^{k+1})$  and  $dN_2^{k+1} = \tau(dN_1^{k+1})$ . For any linear map  $\kappa: N_1^{k+1} \rightarrow \mathcal{Z}_2^{k+1}$ , we can define an isomorphism  $\phi_{\kappa}: \mathcal{E}_1 \rightarrow \mathcal{E}_2$  by setting it to equal  $\tau$  on  $\mathcal{E}_1^{\leq k}$ ,  $\tau_{\#}$  on  $C_1^{k+1}$ , and  $\kappa + d^{-1} \circ \tau \circ d$  on  $N_1^{k+1}$  (taking the inverse of  $d: N_2^{k+1} \rightarrow dN_2^{k+1}$ )—indeed any isomorphism  $\phi: \mathcal{E}_1 \rightarrow \mathcal{E}_2$  such that  $\phi^{\leq k} = \tau$  is of this form.

It remains to understand  $\phi_{\kappa}^* \alpha_2$ . In the decomposition from (ii) of  $H^m(\mathcal{E}_1)$  as the direct sum of the image of  $H^m(\mathcal{E}_1^{\leq k})$ ,  $C_1^{k+1} \otimes H^{m-k-1}(\mathcal{E}_1^{\leq k})$  and  $W$ , the restrictions of  $\alpha_1$  and  $\phi_{\kappa}^* \alpha_2$  to the first two summands agree for any  $\kappa$ .

Let  $W \subseteq \mathcal{E}_1^m$  be a subspace of closed representatives of  $W$ . The projection  $p: \mathcal{E}_1^m \rightarrow N^{k+1} \otimes \mathcal{N}_1^{m-k-1}$  (with kernel  $\mathcal{N}_1^m \oplus C_1^{k+1} \otimes \mathcal{N}_1^{m-k-1}$ ) maps  $W \hookrightarrow N_1^{k+1} \otimes \mathcal{Z}_1^{m-k-1}$ . Let us now explain that the induced map  $p: W \rightarrow N_1^{k+1} \otimes H^{m-k-1}(\mathcal{N}_1)$  is injective too.

Suppose that for some  $w \in W$ ,  $p(w) = \sum n_j \otimes dx_j$ , for  $n_j \in N_1^{k+1}$  and  $x_j \in \mathcal{N}_1^{m-k-2}$ . Note that  $\sum dn_j \otimes x_j \in \mathcal{N}_1^{k+2} \mathcal{N}_1^{m-k-2} \subseteq \mathcal{N}_1^m$ . Therefore  $w - d(\sum n_j \otimes x_j)$ , which represents the same class in  $H^m(\mathcal{N}_1)$  as  $w$ , lies in the kernel of  $p$ . But  $W$  is by definition as a direct complement to the space of classes represented by elements of the kernel of  $p$ . Hence  $W \hookrightarrow N_1^{k+1} \otimes H^{m-k-1}(\mathcal{N}_1)$ .

The restriction of  $\phi_{\kappa}^* \alpha_2 - \phi_0^* \alpha_2$  to  $W$  equals the composition of  $p$  with the linear map  $N_1^{k+1} \otimes H^{m-k-1}(\mathcal{N}_1) \rightarrow \mathbb{Q}$ ,  $n \otimes x \mapsto \alpha_2(\kappa(n)\tau_{\#}x)$ . Because  $\tau_{\#}: H^{m-k-1}(\mathcal{N}_1) \rightarrow H^{m-k-1}(\mathcal{N}_2)$  and  $\alpha_2 \cap: H^{m-k-1}(\mathcal{N}_2) \rightarrow H^{k+1}(\mathcal{N}_2)^*$  are isomorphisms, any linear functional on  $N_1^{k+1} \otimes H^{m-k-1}(\mathcal{N}_1)$  is realised this way for some  $\kappa: N_1^{k+1} \rightarrow \mathcal{Z}_2^{k+1}$ . Thus by adjusting the choice of  $\kappa$ , the restriction of  $\phi_{\kappa}^* \alpha_2$  to  $W$  can be made to equal  $\alpha_1$ .  $\square$

**Lemma 3.4.** (i) Let  $\mathcal{N}$  be a  $(n-1)$ -connected minimal Sullivan algebra generated in degree  $< 2n$ . Then  $\mathcal{F}: \mathcal{B}_n(\mathcal{N}) \rightarrow H^{4n-1}(\mathcal{N})$  is injective, and its image has trivial intersection with the image of the linear map  $(\Lambda H^{<2n}(\mathcal{N}))^{4n-1} \rightarrow H^{4n-1}(\mathcal{N})$  induced by the multiplication.

(ii) Let  $\mathcal{N}_1$  and  $\mathcal{N}_2$  be  $(n-1)$ -connected minimal Sullivan algebras generated in degree  $< 2n$ . Given an isomorphism  $G: H^{<2n}(\mathcal{N}_1) \rightarrow H^{<2n}(\mathcal{N}_2)$  of the truncated cohomology rings, and  $(2n-1)$ -partial Poincaré classes  $\alpha_j \in H^{4n-1}(\mathcal{N}_j)^*$ , there is an isomorphism  $\tau: \mathcal{N}_1 \rightarrow \mathcal{N}_2$  such that  $\tau_{\#} = G$  on  $H^{<2n}(\mathcal{N}_1)$  and  $\tau_{\#}^* \alpha_2 = \alpha_1$  if and only if the diagram below commutes.

$$\begin{array}{ccc} (\Lambda H^{<2n}(\mathcal{N}_1))^{4n-1} \oplus \mathcal{B}_n(\mathcal{N}_1) & \longrightarrow & H^{4n-1}(\mathcal{N}_1) \xrightarrow{\alpha_1} \mathbb{Q} \\ \downarrow G & & \downarrow \\ (\Lambda H^{<2n}(\mathcal{N}_2))^{4n-1} \oplus \mathcal{B}_n(\mathcal{N}_2) & \longrightarrow & H^{4n-1}(\mathcal{N}_2) \xrightarrow{\alpha_2} \mathbb{Q} \end{array}$$

*Proof.* The argument is similar to the induction steps in the proofs of Proposition 3.3(ii) and (iii).

(i) By a trivial recursion we find that the generating space for  $\mathcal{N} = \Lambda V$  has  $V^i = 0$  for  $0 < i < n$ ,  $V^i = C^i$  for  $n \leq i \leq 2n-2$ , while  $V^{2n-1} = C^{2n-1} \oplus N^{2n-1}$ , with  $C^i \cong H^i(\mathcal{N})$  and  $d$  mapping  $N^{2n-1}$  isomorphically to  $\ker(\mathcal{G}_\epsilon^2 C^n \rightarrow H^{2n}(\mathcal{N})) \cong E_n$ . Hence

$$\begin{aligned} \mathcal{N}^{4n-1} &= (\Lambda C)^{4n-1} \oplus N^{2n-1} \otimes \mathcal{G}_\epsilon^2 C^n, \\ \mathcal{N}^{4n-2} &= (\Lambda C)^{4n-2} \oplus \Lambda^2 N^{2n-1} \oplus C^{2n-1} \otimes N^{2n-1}. \end{aligned}$$

Let  $K$  be the closed subspace of the second term of  $\mathcal{N}^{4n-1}$ . Then

$$H^{4n-1}(\mathcal{N}) = \frac{(\Lambda C)^{4n-1}}{C^{2n-1} \otimes dN^{2n-1}} \oplus \frac{K}{d\Lambda^2 N^{2n-1}}, \quad (13)$$

and the first term is clearly the image of  $(\Lambda H^{<2n})^{4n-1}$ . The isomorphism  $N^{2n-1} \otimes \mathcal{G}_\epsilon^2 C^n \cong E_n \otimes \mathcal{G}_\epsilon^2 H^n(M)$  maps  $d\Lambda^2 N^{2n-1} \cong \text{Alt}^2 E_n$  and  $K$  to the kernel of  $E_n \otimes \mathcal{G}_\epsilon^2 H^n(M) \rightarrow \mathcal{G}_\epsilon^4 H^n(M)$ . If we let  $Z$  be the intersection with  $E_n \otimes E_n$  and pick a direct complement  $W$  to  $E_n$  in  $\mathcal{G}_\epsilon^2 H^n(\mathcal{N})$  (so  $W \cong H^{2n}(\mathcal{N})$ ) then  $K \cong Z \oplus E_n \otimes W$ . Since  $Z/\text{Alt}^2 E_n \cong \mathcal{B}_\epsilon[E_n] = \mathcal{B}_n(\mathcal{N})$ , the second term in (13) is  $\mathcal{B}_n(\mathcal{N}) \oplus W$ , and  $\mathcal{F}$  maps isomorphically to the first summand (*cf.* Remark 2.2).

(ii) The restriction of  $\tau$  to  $\Lambda C_1 \subseteq \mathcal{N}_1$  is determined by  $G$ . Thus the only flexibility that remains for adjusting  $\tau_{\#}^* \alpha_2$  is the restriction of  $\tau$  to  $N_1^{2n-1}$ .

The above computation shows that if we take  $\mathcal{W}$  to be a direct complement of  $dN_1^{2n-1}$  in  $\mathcal{G}_\epsilon^2 C_1^n$ , then  $N_1^{2n-1} \otimes \mathcal{W} \subseteq \mathcal{N}_1^{4n-1}$  represents a direct complement of the image of  $(\Lambda H^{<2n}(\mathcal{N}_1))^{4n-1} \oplus \mathcal{B}_n(\mathcal{N}_1)$  in  $H^{4n-1}(\mathcal{N}_1)$ . Adding a linear map  $\kappa: N_1^{2n-1} \rightarrow C_2^{2n-1}$  to  $\tau|_{N_1^{2n-1}}$  changes the restriction of  $\tau_{\#}^* \alpha_2$  to  $N_1^{2n-1} \otimes \mathcal{W}$  by adding  $(x, q) \mapsto \alpha_2(\kappa(x)G(q))$  (while leaving the restriction to the image of  $(\Lambda H^{<2n}(\mathcal{N}_1))^{4n-1} \oplus \mathcal{B}_n(\mathcal{N}_1)$  unchanged, as it must). Since  $G: \mathcal{W} \hookrightarrow H^{2n}(\mathcal{N}_2)$  while  $\alpha_2 \cap: H^{2n}(\mathcal{N}_2) \hookrightarrow H^{2n-1}(\mathcal{N}_2)^*$  by the hypothesis that  $\alpha_2$  is a  $(2n-1)$ -partial Poincaré class, any map  $N_1^{2n-1} \otimes \mathcal{W} \rightarrow \mathbb{Q}$  is realised this way for some  $\kappa: N_1^{2n-1} \rightarrow C_2^{2n-1}$ .  $\square$

*Proof of Theorem 3.2.* (i) Up to isomorphism there is a unique minimal algebra  $\mathcal{N}$  generated in degree  $< 2n$  with  $H^i(\mathcal{N}) \cong H^i$  for  $i < 2n$  and  $H^{2n}(\mathcal{N}) \hookrightarrow H^{2n}$ . By Lemma 3.4(i) we may choose  $\alpha: H^{4n-1}(\mathcal{N}) \rightarrow \mathbb{Q} \cong H^{4n-1}$  so that the diagram below commutes.

$$\begin{array}{ccc} (\Lambda H^{<2n}(\mathcal{N}))^{4n-1} \oplus \mathcal{B}_n(\mathcal{N}) & \longrightarrow & H^{4n-1}(\mathcal{N}) \\ \downarrow & & \downarrow \alpha \\ (\Lambda H^{<2n})^{4n-1} \oplus \mathcal{B}_n(H^*) & \longrightarrow & H^{4n-1} \end{array}$$

This  $\alpha$  is a  $(2n-1)$ -partial Poincaré class, and we may apply Proposition 3.3(ii) to construct the desired  $\mathcal{M}$ .

(ii) Let  $\alpha_j \in H^{4n-1}(\mathcal{M}_j)$  be the Poincaré classes. Their pull-backs to  $H^{4n-1}(\mathcal{M}_j^{<2n})$  are  $(2n-1)$ -partial Poincaré classes. The restriction to the image of  $(\Lambda H^{<2n}(\mathcal{M}_j^{<2n}))^{4n-1}$  is determined by

the algebra structure of  $H^*(\mathcal{M}_j)$ , while the restriction to  $\mathcal{B}_n(\mathcal{M}_j)$  are determined by the Bianchi-Massey tensor. Therefore we may apply Lemma 3.4(ii) to find an isomorphism  $\mathcal{M}_1^{<2n} \cong \mathcal{M}_2^{<2n}$ , which can then be extended to the desired isomorphism  $\mathcal{M}_1 \cong \mathcal{M}_2$  by Proposition 3.3(iii).  $\square$

**3.2. Minimal models and manifolds.** A *minimal model* of a DGA  $\mathcal{A}$  is a minimal Sullivan algebra  $\mathcal{M}$  together with a quasi-isomorphism  $q : \mathcal{M} \rightarrow \mathcal{A}$ , i.e. a DGA homomorphism whose induced map  $q_\# : H^*(\mathcal{M}) \rightarrow H^*(\mathcal{A})$  is an isomorphism.

Recall that a minimal model of a topological space  $X$  is a minimal model of  $\Omega_{\text{PL}}(X)$  and that every quasi-isomorphism of minimal models is realised by a rational homotopy equivalence of spaces, see [22, §8] and [11, Proposition 17.13]. Theorem 1.2 now follows directly from the following claim.

**Corollary 3.5.** *Let  $\mathcal{A}$  and  $\mathcal{A}'$  be  $(n-1)$ -connected  $(4n-1)$ -dimensional Poincaré DGAs ( $n \geq 2$ ), with minimal models  $q : \mathcal{M} \rightarrow \mathcal{A}$  and  $q' : \mathcal{M}' \rightarrow \mathcal{A}'$ . If  $G : H^*(\mathcal{A}) \rightarrow H^*(\mathcal{A}')$  is an isomorphism of the cohomology rings then there exists a DGA isomorphism  $\phi : \mathcal{M} \rightarrow \mathcal{M}'$  such that  $q_\# \circ \phi_\# = G \circ q'_\#$  if and only if the diagram below commutes.*

$$\begin{array}{ccc} \mathcal{B}_n(\mathcal{A}) & \xrightarrow{G} & \mathcal{B}_n(\mathcal{A}') \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F}' \\ H^{4n-1}(\mathcal{A}) & \xrightarrow{G} & H^{4n-1}(\mathcal{A}') \end{array}$$

*Proof.* Apply Theorem 3.2(ii) to  $(q'_\#)^{-1} \circ G \circ q_\#$ .  $\square$

Next recall that a DGA  $\mathcal{A}$  is said to be *formal* if there is a quasi-isomorphism  $\widehat{q} : \mathcal{M} \rightarrow (H^*(\mathcal{A}), 0)$  from its minimal model  $\mathcal{M}$ —in other words, if  $\mathcal{A}$  and  $(H^*(\mathcal{A}), 0)$  have the same minimal model. A space  $X$  is formal if its DGA of piecewise linear de Rham forms is. The following proposition is therefore the algebraic formulation of Theorem 1.3.

**Corollary 3.6.** *An  $(n-1)$ -connected  $(4n-1)$ -dimensional Poincaré DGA  $\mathcal{A}$  is formal if and only if the Bianchi-Massey tensor  $\mathcal{F} : \mathcal{B}_n(\mathcal{A}) \rightarrow H^{4n-1}(\mathcal{A})$  is trivial.*

*Proof.* Since the DGA  $(H^*(\mathcal{A}), 0)$  has  $\mathcal{F} = 0$ , the functoriality of the Bianchi-Massey tensor implies that if  $\mathcal{A}$  is formal then its minimal model, and hence  $\mathcal{A}$  itself, also have  $\mathcal{F} = 0$ .

Conversely if  $\mathcal{F} = 0$  then we can let  $\mathcal{A}' := (H^*(\mathcal{A}), 0)$ , set  $G : H^*(\mathcal{A}) \rightarrow H^*(\mathcal{A}')$  to be the tautological isomorphism and apply Corollary 3.5 to deduce that  $\mathcal{A}$  is formal.  $\square$

*Remark 3.7.* Our reasoning has been guided by the notion of  $k$ -formality of Fernández and Muñoz [12]. They define  $M$  to be  $k$ -formal if one can choose the generating set  $V$  for a minimal model  $\mathcal{M}$  and direct complements  $N^i$  to  $C^i \subseteq V^i$  for  $i \leq k$  so that the cohomology of the ideal  $I_k := N^{\leq k} \mathcal{M}^{\leq k} \subseteq \mathcal{M}^{\leq k}$  maps trivially to  $H^*(M)$ . According to [12, Theorem 3.1], a closed orientable manifold of dimension  $\leq 2k+1$  is formal if and only if it is  $k$ -formal—this can also be deduced from Proposition 3.3 (under the simplifying assumption of simple-connectedness).

The algebraic considerations in Lemma 3.4 essentially identify the Bianchi-Massey tensor as a complete obstruction to  $(2n-1)$ -formality for closed  $(n-1)$ -connected  $(4n-1)$ -manifolds. One could thus prove Theorem 1.3 more briefly by appealing to the results of Fernández and Muñoz, but we have set up the argument to prove the more general Theorem 1.2 too.

We now move from rational classification to the classification of manifolds up to finite ambiguity. Kreck and Triantafyllou [18] define an invariant  $\alpha_M$  of a closed manifold  $M$ . In the case of a  $(4n-1)$ -manifold  $M$ ,

$$\alpha_M \in H^{4n-1}(\mathcal{M}^{<2n})^*$$

is defined by composing the natural map  $H^{4n-1}(\mathcal{M}^{<2n}) \rightarrow H^{4n-1}(M)$  with  $\int_M : H^{4n-1}(M) \rightarrow \mathbb{Q}$ . According to [18, Theorem 2.2], the diffeomorphism type of  $M$  with formal  $(2n+1)$ -skeleton is determined up to finite ambiguity by the truncated cohomology ring  $\oplus_{i \leq 2n+1} H^i(M; \mathbb{Z})$ , the real Pontrjagin classes and  $\alpha_M$ . If  $M$  is  $(n-1)$ -connected then the  $(2n+1)$ -skeleton is certainly formal. Theorem 1.8, that the full integral cohomology ring, Pontrjagin classes and Bianchi-Massey tensor classify  $M$  up to finite ambiguity, is thus an immediate consequence of the following claim.

**Corollary 3.8.** *Let  $M$  be a closed  $(n-1)$ -connected  $(4n-1)$ -manifold. Then  $\alpha_M$  is determined by the rational cohomology ring  $H^*(M)$  and the Bianchi-Massey tensor  $\mathcal{F}: \mathcal{B}_n(M) \rightarrow H^{4n-1}(M)$ .*

*Proof.* Immediate from Lemma 3.4.  $\square$

**3.3. Rational realisation.** In this subsection we prove Theorem 1.5 using Theorem 3.2(i) and by making small adjustments to Sullivan's proof of [22, Theorem 13.2].

We are given  $(H^*, p_*, \mathcal{F})$  a  $(n-1)$ -connected  $(4n-1)$ -dimensional rational Poincaré duality algebra  $H^*$  with candidate Pontrjagin classes  $p_* \in H^{4*}$  and Bianchi-Massey tensor  $\mathcal{F}: \mathcal{B}_n(H^*) \rightarrow H^{4n-1}$ . By Theorem 3.2(i) there is a Sullivan minimal algebra  $\mathcal{M}$  with  $H^*(\mathcal{M}) = H^*$  and Bianchi-Massey tensor  $\mathcal{F}$ . By [22, §8] (see also [11, Theorem 17.10]),  $\mathcal{M}$  is realised by a rational space  $X$  which is  $(n-1)$ -connected since  $\mathcal{M}$  is  $(n-1)$ -connected. The cohomology classes  $p_* \in H^{4*}(\mathcal{M}) = H^{4*}(X)$  define a map  $p: X \rightarrow \Pi_{4i \geq n} K(\mathbb{Q}, 4i)$  to the indicated product of rational Eilenberg-MacLane spaces. If  $BO\langle n \rangle$  denotes the  $(n-1)$ -connected cover of  $BO$ , then the universal Pontrjagin classes on  $BO\langle n \rangle$  define a rational equivalence  $p\langle n \rangle: BO\langle n \rangle \rightarrow \Pi_{4i \geq n} K(\mathbb{Q}, 4i)$  and we let  $Y$  be the rational space in the following pullback square:

$$\begin{array}{ccc} Y & \longrightarrow & BO\langle n \rangle \\ \downarrow & & \downarrow p\langle n \rangle \\ X & \xrightarrow{p} & \Pi_{4i \geq n} K(\mathbb{Q}, 4i) \end{array}$$

We note that  $Y \rightarrow X$  is a rational equivalence since  $p\langle n \rangle$  is a rational equivalence. If we set  $T$  to be the Thom space of the stable bundle over  $Y$  induced by  $Y \rightarrow BO\langle n \rangle$ , then the stable homotopy groups of  $T$  satisfy  $\pi_{4n-1}^S(T) \otimes \mathbb{Q} \cong H_{4n-1}(Y) \cong H_{4n-1}(X) = \mathbb{Q}$  since  $Y \rightarrow X$  is a rational equivalence. Hence there is a closed smooth  $(4n-1)$ -manifold  $M$  and a normal map  $f: M \rightarrow Y$  of non-zero degree. Since  $Y$  is  $(n-1)$ -connected, we may perform surgery below the middle dimension [27, §1] on  $f: M \rightarrow Y$  to make  $M$   $(n-1)$ -connected. We continue further with rational surgery as in the proof of [22, Theorem 13.2] to achieve that  $f: M \rightarrow Y$  is a rational equivalence with  $M$  still  $(n-1)$ -connected. Then  $(H^*(M), p_*(M), \mathcal{F}(M)) = (H^*(X), p_*, \mathcal{F})$ , proving Theorem 1.5.

#### 4. COBOUNDARIES AND INTEGRALITY

In this section we show how to compute the Bianchi-Massey tensor of  $M$  if  $M$  has a coboundary  $W$  such that the restriction homomorphism  $H^n(W) \rightarrow H^n(M)$  is onto. We can use this to construct compact manifolds  $W$  whose boundaries  $M$  realise a given cup-product structure and Bianchi-Massey tensor.

**4.1. Computing the Bianchi-Massey tensor via a coboundary.** Let  $W$  be a compact  $4n$ -manifold with boundary  $M$ , such that the restriction  $j: H^n(W) \rightarrow H^n(M)$  is surjective. Pick a right inverse  $r$  of  $j$ . We will denote it by  $x \mapsto \hat{x}$ , and also write the induced map  $\mathcal{G}_e^2 H^n(M) \rightarrow H^{2n}(W)$  as  $e \mapsto \hat{e}$ .

Note that if  $e \in E_n$  then  $\hat{e}$  belongs to  $H_0^{2n}(W) \subseteq H^{2n}(W)$ , the image of  $H^{2n}(W, M)$ . The intersection form  $\lambda_W: H_0^{2n}(W) \times H_0^{2n}(W) \rightarrow \mathbb{Q}$  is well-defined.

**Lemma 4.1.** *Let  $A_W: P^2 E_n \rightarrow \mathbb{Q}$  be induced by  $e e' \mapsto -\lambda_W(\hat{e}, \hat{e}')$ . The restriction of  $A_W$  to  $\mathcal{B}_n(M)$  equals the Bianchi-Massey tensor  $\mathcal{F}_M$ .*

*Proof.* Choose  $\hat{\alpha}: H^n(M) \rightarrow \Omega_{\text{PL}}^n(W)$  so that

$$\begin{array}{ccc} H^n(M) & \xhookrightarrow{r} & H^n(W) \\ \alpha \downarrow & \searrow \hat{\alpha} & \uparrow \\ \Omega_{\text{PL}}^n(M) & \xleftarrow{\quad} & \Omega_{\text{PL}}^n(W) \end{array}$$

commutes. Then  $\hat{\alpha}^2(e), \hat{\alpha}^2(e') \in \Omega_{\text{PL}}^{2n}(W)$  are representatives of  $\hat{e}$  and  $\hat{e}'$ . If  $d\gamma(e) = \alpha^2(e)$  as in the definition of  $\mathcal{F}$  and  $\rho: W \rightarrow [0, 1]$  is a cut-off function supported on a collar neighbourhood of  $M$

then  $\hat{\alpha}^2(e) - d(\rho\gamma(e)) \in \Omega_{\text{PL}}^{2n}(W)$  represents a pre-image of  $u \in H^{2n}(W, M)$  of  $\hat{e}$ , and by definition  $\lambda_W(\hat{e}, \hat{e}') = u \hat{w}$ . In the notation of (5), we can write this as

$$-A_W = \int_W (\hat{\alpha}^2 - d(\rho\gamma)) \cdot \hat{\alpha}$$

Hence, as maps  $P^2E_n \rightarrow \mathbb{Q}$ ,

$$\int_M \gamma \cdot \alpha^2 = \int_W d(\rho\gamma \cdot \hat{\alpha}^2) = A_W - \int_W \hat{\alpha}^2 \cdot \hat{\alpha}^2.$$

The last term factors through  $P^2E_n \rightarrow \mathcal{G}_\epsilon^4 H^n(M)$ , so vanishes on  $\mathcal{B}_n(M)$ , while the restriction of the left hand side to  $\mathcal{B}_n(M)$  equals  $\mathcal{F}_M$  by definition.  $\square$

By (11) this also lets us compute Massey triple products using the coboundary  $W$ , so the lemma is a generalisation of [15, Proposition 3.2.6].

If the cup product  $H^n(M) \times H^n(M) \rightarrow H^{2n}(M)$  is trivial (so  $E_n = \mathcal{G}_\epsilon^2 H^n(M)$ ) then the cup-square  $\mathcal{G}_\epsilon^2 H^n(W) \rightarrow H_0^{2n}(W)$  and the intersection form (together with  $r$ ) define an element of  $\text{Sym}^2 \text{Grad}_\epsilon^2 H^n(M)^*$ . In view of Remark 2.4, the lemma means that the Bianchi-Massey tensor measures the failure of that 4-tensor to be fully  $\epsilon$ -symmetric. More generally, when  $M$  bounds over  $H^n(M)$  we could use Lemma 4.1 as the definition of the Bianchi-Massey tensor, and deduce that it is independent of the choice of coboundary from the full  $\epsilon$ -symmetry of the quadruple cup product on  $H^n$  of a closed  $4n$ -manifold.

The coboundary perspective is useful for understanding the relation between the Bianchi-Massey tensor and cohomology with integer coefficients. Recall that  $\bar{c} : \mathcal{G}_\epsilon^2 H^n(M; \mathbb{Z}) \rightarrow H^{2n}(M; \mathbb{Z})$  is the integral cup-square map. Let  $\bar{E}_n$  denote the kernel of  $\bar{c}$  modulo torsion (or equivalently, the pre-image of  $E_n$  under the map  $\mathcal{G}_\epsilon^2 H^n(M; \mathbb{Z}) \rightarrow \mathcal{G}_\epsilon^2 H^n(M)$ ), and  $\mathcal{B}_n(M; \mathbb{Z}) = \mathcal{B}_\epsilon[\bar{E}_n]$  the kernel of  $P^2\bar{E}_n \rightarrow \mathcal{G}_\epsilon^4 H^n(M; \mathbb{Z})$ . While it is hard to see how to define an integral version of the Bianchi-Massey tensor in terms of singular cochains, we may obviously define the “integral restriction”  $\bar{\mathcal{F}}_M : \mathcal{B}_n(M; \mathbb{Z}) \rightarrow \mathbb{Q}$  as the composition of  $\mathcal{F}_M$  with  $\mathcal{B}_n(M; \mathbb{Z}) \rightarrow \mathcal{B}_n(M)$ , as we did in the introduction.

If  $M = \partial W$  and  $H^n(W; \mathbb{Z}) \rightarrow H^n(M; \mathbb{Z})$  is onto, we shall say that  $W$  is a coboundary over  $H^n(M; \mathbb{Z})$ . In this case the Bianchi-Massey tensor of  $M$  is related to the torsion linking form of  $M$

$$b_M : TH^{2n}(M; \mathbb{Z}) \times TH^{2n}(M; \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

To describe this relationship we let  $B_M : P^2\bar{E}_n \rightarrow \mathbb{Q}/\mathbb{Z}$  be the homomorphism induced by

$$\bar{E}_n \times \bar{E}_n \rightarrow \mathbb{Q}/\mathbb{Z}, (e, e') \mapsto b_M(\bar{c}(e), \bar{c}(e')). \quad (14)$$

**Corollary 4.2.** *Let  $W$  be a closed  $4n$ -manifold with boundary  $M$ , which is a coboundary over  $H^n(M; \mathbb{Z})$ . Pick a right inverse  $\bar{r} : H^n(M; \mathbb{Z}) \rightarrow H^n(W; \mathbb{Z})$ , and define  $\bar{A}_W : P^2\bar{E} \rightarrow \mathbb{Q}$  analogously to  $A$  in Lemma 4.1. Then*

$$\bar{A}_W = B_M \mod \mathbb{Z}$$

and

$$\bar{\mathcal{F}}_M = B_M|_{\mathcal{B}_n(M; \mathbb{Z})} \mod \mathbb{Z}.$$

*Proof.* For the first equality we recall that if  $\bar{x}, \bar{y} \in H^{2n}(W; \mathbb{Z})$  restrict to torsion classes  $x, y \in H^4(M; \mathbb{Z})$ , then by [1, Theorem 2.1]

$$b_M(x, y) = -\lambda_W(\bar{x}, \bar{y}). \quad (15)$$

The second equality follows immediately from the first together with Lemma 4.1.  $\square$

*Remark 4.3.* If  $W$  is a coboundary over  $H^n(M; \mathbb{Z})$  for  $M = \partial W$ , then the mod  $\mathbb{Z}$  reduction of  $\bar{\mathcal{F}}_M : \mathcal{B}_n(M; \mathbb{Z}) \rightarrow \mathbb{Q}$  is determined by the torsion linking form—in particular, if  $H^{2n}(M; \mathbb{Z})$  is torsion-free then  $\bar{\mathcal{F}}$  takes integer values. Because of the non-commutativity of the cup product on singular cochains, we do not see a reason for this claim to be true in the absence of such a coboundary.

**4.2. Realising Bianchi-Massey tensors.** In this subsection we construct  $(n-1)$ -connected  $4n$ -manifolds  $W$  whose boundaries  $M$  realise a large class of Bianchi-Massey tensors. We focus on the following basic invariants of  $M$ :

- (i) the cup square,  $\bar{c}: \mathcal{G}_\epsilon^2 H^n(M; \mathbb{Z}) \rightarrow H^{2n}(M; \mathbb{Z})$ ;
- (ii) the linking form,  $b_M: TH^{2n}(M; \mathbb{Z}) \times TH^{2n}(M; \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$ ;
- (iii) the Bianchi-Massey tensor,  $\bar{\mathcal{F}}_M: \mathcal{B}_n(M; \mathbb{Z}) \rightarrow \mathbb{Q}$ .

Hence we define a *linking model* as an algebraic model of  $M$ , which is a quintuple

$$(F, G, b, \bar{c}, \bar{\mathcal{F}}).$$

Here  $F$  is free abelian,  $G$  is abelian with torsion subgroup  $T \subseteq G$ ,  $b: T \times T \rightarrow \mathbb{Q}/\mathbb{Z}$  is a nonsingular symmetric torsion form, and  $\bar{c}: \mathcal{G}_\epsilon^2 F \rightarrow G$  and  $\bar{\mathcal{F}}: \mathcal{B}_\epsilon[\bar{E}] \rightarrow \mathbb{Q}$  are homomorphisms where for  $\bar{\mathcal{F}}$  we have  $\bar{E} := \text{Ker}(\rho \circ \bar{c})$  for  $\rho: G \rightarrow G/T$  the projection. Recalling Corollary 4.2, we say that  $\bar{\mathcal{F}}$  and  $b$  are *compatible* if

$$\bar{\mathcal{F}} = B|_{\mathcal{B}_\epsilon[\bar{E}]} \pmod{\mathbb{Z}}, \quad (16)$$

where  $B: P^2 \bar{E} \rightarrow \mathbb{Q}/\mathbb{Z}$  is defined from  $b$  as in (14). Our main realisation result is the following

**Theorem 4.4.** *Let  $(F, G, b, \bar{c}, \bar{\mathcal{F}})$  be a linking model with compatible  $b$  and  $\mathcal{F}$ . For all  $n \geq 2$  there exists some  $(n-1)$ -connected closed  $M^{4n-1}$  with isomorphisms  $H^n(M; \mathbb{Z}) \cong F$  and  $H^{2n}(M; \mathbb{Z}) \cong G$  which identify  $(b_M, \bar{c}_M, \bar{\mathcal{F}}_M) = (b, \bar{c}, \bar{\mathcal{F}})$ . In addition, we may assume that  $H^*(M)$  is concentrated in degrees  $*$  = 0,  $n$ ,  $3n-1$  and  $4n-1$  and the same holds for  $H^*(M; \mathbb{Z})$  when  $n = 2, 4$  or  $n$  is odd.*

Before proving Theorem 4.4 we show how it implies Theorem 1.7 of the introduction, where we used simpler algebraic models for  $M$ . A *torsion free model* is a quadruple

$$(F, G, \bar{c}, \bar{\mathcal{F}}),$$

where  $F$  and  $G$  are free abelian and  $\bar{c}: \mathcal{G}_\epsilon^2 F \rightarrow G$  and  $\bar{\mathcal{F}}: \mathcal{B}_\epsilon[\bar{E}] \rightarrow \mathbb{Q}$  are homomorphisms as in a linking model. With this terminology, Theorem 1.7 states that any torsion free model can be realised by a  $(n-1)$ -connected  $(4n-1)$ -manifold. Given a linking model  $(F, G, b, \bar{c}, \bar{\mathcal{F}})$ , we obtain a torsion free model  $(F, G/T, \rho \circ \bar{c}, \bar{\mathcal{F}})$ , so Theorem 1.7 follows immediately from Theorem 4.4 and the following

**Lemma 4.5.** *Given any torsion free model  $(F, G, \bar{c}, \bar{\mathcal{F}})$  there is a linking model  $(F, G + T, b, \bar{c}', \bar{\mathcal{F}})$  with compatible  $b$  and  $\mathcal{F}$  such that  $\bar{c} = \rho \circ \bar{c}'$ .*

*Proof.* Choose any extension of  $\bar{\mathcal{F}}: \mathcal{B}_\epsilon[\bar{E}] \rightarrow \mathbb{Q}$  to  $\bar{\mathcal{F}}': P^2 \bar{E} \rightarrow \mathbb{Q}$  and let  $B_0: P^2 \bar{E} \rightarrow \mathbb{Q}/\mathbb{Z}$  be the composition of  $\bar{\mathcal{F}}'$  with the canonical surjection  $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ . We define  $R \subseteq \bar{E}$  to be the radical of  $B_0$ :

$$R := \{r \in \bar{E} \mid B_0(re) = 0 \ \forall e \in \bar{E}\}.$$

We let  $S \subseteq \bar{E}/R$  be the torsion subgroup and fix a projection  $\bar{E}/R \rightarrow S$ , which in turn defines a surjection  $\pi: \bar{E} \rightarrow S$ . The map  $\pi$  induces a surjection  $P^2(\pi): P^2 \bar{E} \rightarrow P^2 S$  such that  $B_0$  factors over  $P^2(\pi)$ ; i.e.  $B_0$  induces a (possibly singular) symmetric torsion form

$$b_0: P^2 S \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Set  $\hat{S} := \text{Hom}(S, \mathbb{Q}/\mathbb{Z})$  and define a nonsingular torsion form  $b$  by

$$b: (S \times \hat{S}) \times (S \times \hat{S}) \rightarrow \mathbb{Q}/\mathbb{Z}, \quad b((s_1, \sigma_1), (s_2, \sigma_2)) = b_0(s_1, s_2) + \sigma_1(s_2) + \sigma_2(s_1).$$

To finish the proof, let  $p: \mathcal{G}_\epsilon^2 F \rightarrow \bar{E}$  be any projection, set  $(T, b) := (S \oplus \hat{S}, b)$  and define  $\bar{c}'$  by

$$\bar{c}': \mathcal{G}_\epsilon^2 F \rightarrow G \oplus T, \quad ff' \mapsto (\bar{c}(ff'), \pi \circ p(ff')).$$

It is clear that  $\bar{c} = \rho \circ \bar{c}'$  and  $b$  and that  $\bar{\mathcal{F}}$  and  $b$  are compatible; i.e. satisfy (16).  $\square$

To prove Theorem 4.4, we note that it follows directly from the next two lemmas. Lemma 4.6 is a generalisation of results of Schmitt [20] and we defer its proof to the end of the section. We will use  $-\vee$  to denote  $\text{Hom}(-, \mathbb{Z})$ , while  $-^*$  denotes duals of vector spaces.



**Lemma 4.6.** *Let free abelian groups  $F$  and  $G_1$ , a homomorphism  $\bar{c}_1: \mathcal{G}_\epsilon^2 F \rightarrow G_1$  and an even symmetric bilinear form  $\lambda_1: G_1 \times G_1 \rightarrow \mathbb{Z}$  form be given. Then for all  $n \geq 2$  there is a compact  $(n-1)$ -connected  $4n$ -manifold  $W = W(F, G_1, \bar{c}_1, \lambda_1)$  with  $(n-1)$ -connected boundary such that the following hold:*

- (i)  $W$  has the homotopy type of a  $(n-1)$ -connected  $4n$ -dimensional CW complex;
- (ii)  $H^*(W)$  is concentrated in dimensions  $0, n$  and  $2n$ ;
- (iii)  $H^*(W; \mathbb{Z})$  is concentrated in dimensions  $0, n$  and  $2n$  when  $n = 2, 4$  or  $n$  is odd;
- (iv)  $H^n(W; \mathbb{Z}) = F$ ;
- (v)  $H^{2n}(W; \mathbb{Z}) = G_1$ ;
- (vi)  $\bar{c}_W = \bar{c}_1: \mathcal{G}_\epsilon^2 H^n(W; \mathbb{Z}) \rightarrow H^{2n}(W; \mathbb{Z})$ ;
- (vii)  $\lambda_W = \lambda_1$ .

**Lemma 4.7.** *Given a linking model  $(F, G, b, \bar{c}, \bar{\mathcal{F}})$ , we can find  $(G_1, \bar{c}_1, \lambda_1)$  such that the manifold  $W(F, G_1, \bar{c}_1, \lambda_1)$  of Lemma 4.6 has boundary  $M$  with*

$$(H^n(M; \mathbb{Z}), H^{2n}(M; \mathbb{Z}), b_M, \bar{c}_M, \bar{\mathcal{F}}_M) \cong (F, G, b, \bar{c}, \bar{\mathcal{F}}).$$

*Proof.* By [25, Theorem 6] there is a nondegenerate symmetric bilinear form on a free abelian group  $G_2$  such that  $\lambda_2: G_2 \times G_2$  presents  $-b$ . That is, there is a surjection  $\pi_2: G_2 \rightarrow T$  such that

$$\lambda_2(y_1, y_2) = -b(\pi_2(y_1), \pi_2(y_2)) \pmod{\mathbb{Z}},$$

for all  $(y_1, y_2) \in G_2 \times G_2$ . We now turn to  $\bar{c}: \mathcal{G}_\epsilon^2 F \rightarrow G$  and recall that  $\bar{E} = \ker(\bar{c} \circ \rho) \subseteq \mathcal{G}_\epsilon^2 F$  is a summand. We fix a projection  $p: \mathcal{G}_\epsilon^2 F \rightarrow \bar{E}$  and notice that  $\bar{c}|_{\bar{E}}: \bar{E} \rightarrow T \subseteq G$ . Since  $\bar{E}$  is free, we can choose a homomorphism  $q: \bar{E} \rightarrow G_2$  such that  $\pi_2 \circ q = \rho \circ \bar{c}|_{\bar{E}}: \bar{E} \rightarrow T$ .

To apply Lemma 4.6, we choose  $(G_1, \bar{c}_1, \lambda_1)$  as follows:

- (i)  $G_1 = G/T \oplus G_2 \oplus \bar{E} \oplus \bar{E}^\vee$ ,
- (ii)  $\bar{c}_1 = (\rho \circ \bar{c}, q \circ p, p, 0)$ ,
- (iii)  $(G_1, \lambda_1) = (G/T, 0) \oplus (G_2, \lambda_2) \oplus (\bar{E} \oplus \bar{E}^\vee, \lambda_3)$ ,

where  $\lambda_3((e_1, \alpha_1), (e_2, \alpha_2)) = \lambda_{\bar{E}}(e_1, e_2) + \alpha_1(e_2) + \alpha_2(e_1)$ , for  $(\bar{E}, \lambda_{\bar{E}})$  an even symmetric bilinear form which we shall vary as needed. From the exact sequence

$$\dots \rightarrow H^{4n}(W, M; \mathbb{Z}) \rightarrow H^{4n}(W; \mathbb{Z}) \rightarrow H^{4n}(M; \mathbb{Z}) \rightarrow H^{4n+1}(W; \mathbb{Z}) \rightarrow \dots$$

and the fact that  $H^{4n+1}(W; \mathbb{Z}) = 0$ , we have  $H^{2n}(M; \mathbb{Z}) = G/T \oplus T = G$ . By (15)  $b_M = b$ . By Lemma 4.6  $\bar{c}_W = \bar{c}_1$ , and so  $\text{Ker}(\bar{c}_W) = \bar{E}$  since it is the intersection of the kernels of  $\bar{c}$ ,  $p$  and  $q \circ p$ . It follows that  $\bar{c}_M: \mathcal{G}_\epsilon^2 H^n(M; \mathbb{Z}) \rightarrow H^{2n}(M; \mathbb{Z})$  is identified with  $\bar{c}$ .

It remains to determine  $\bar{\mathcal{F}}_M$  and we do this using Lemma 4.1. Note that the map  $A$  that computes  $\mathcal{F}_M$  in Lemma 4.1 is simply the map  $P^2 E \rightarrow \mathbb{Q}$  induced by  $\lambda_1$ . It follows that  $\bar{\mathcal{F}}_M = \bar{\mathcal{F}}_2 + \bar{\mathcal{F}}_{\bar{E}}$  where  $\bar{\mathcal{F}}_2$  is induced by  $\lambda_2$  and  $\bar{\mathcal{F}}_{\bar{E}}$  is induced by  $\lambda_{\bar{E}}$ . By construction  $\bar{\mathcal{F}}_M = \bar{\mathcal{F}}_2 \pmod{\mathbb{Z}}$ . Hence it remains to show that  $\lambda_{\bar{E}}$  can be chosen to realise any integer-valued homomorphism  $\bar{\mathcal{F}}: \mathcal{B}_n(M; \mathbb{Z}) \rightarrow \mathbb{Z}$ , and we do this in the following paragraphs.

Letting  $\text{Sym}_0^2$  denote even symmetric bilinear forms, we thus want to prove that the composition of  $\text{Sym}_0^2 \bar{E}^\vee \rightarrow (P^2 \bar{E})^\vee$  with restriction to  $\mathcal{B}_\epsilon[\bar{E}]$  maps onto  $\mathcal{B}_\epsilon[\bar{E}]^\vee$ . Given that there is an isomorphism  $\text{Sym}^2 \text{Grad}_\epsilon^2 H^n(M; \mathbb{Z})^\vee \cong (P^2 \mathcal{G}_\epsilon^2 H^n(M; \mathbb{Z}))^\vee$ , it suffices to prove the surjectivity mod 2.

Now, the annihilator of  $\text{Sym}_0^2 \text{Grad}_\epsilon^2 H^n(M; \mathbb{Z}_2)^*$  in  $P^2 \mathcal{G}_\epsilon^2 H^n(M; \mathbb{Z}_2)$  is the  $\mathbb{Z}_2$ -vector space of squares of elements of  $\mathcal{G}_\epsilon^2 H^n(M; \mathbb{Z}_2)$ . That clearly intersects trivially with  $\mathcal{B}_\epsilon(H^n(M; \mathbb{Z}_2))$ , since expanding the square of a non-zero element  $\mathcal{G}_\epsilon^2 H^n(M; \mathbb{Z}_2)$  to an element of  $\mathcal{G}_\epsilon^4 H^n(M; \mathbb{Z}_2)$  can never give 0.  $\square$

We turn to the proof of Lemma 4.6. Our first step is to identify a finite  $(n-1)$ -connected  $2n$ -dimensional CW complex  $K(\bar{c})$  which realises a given cup product structure  $(F, G_1, \bar{c}_1)$ . For a rank  $r$  free abelian group  $F$ , let  $K(F^\vee, n)$  be the indicated Eilenberg-MacLane space, and  $K(F^\vee, n)^{(2n-2)}$  a  $(2n-2)$ -skeleton of  $K(F^\vee, n)$ . Attach  $b_{2n-2}(K(F^\vee, n)^{(2n-2)})$   $(2n-1)$ -cells to kill  $H^{2n-2}(K(F^\vee, n)^{(2n-2)})$ , calling the resulting complex  $K'_0$ . The space  $K'_0$  has the rational homotopy type of  $\bigvee_{i=1}^r S^n$  and by the standard inductive construction of  $K(F^\vee, n)$  as in [14, Example 4.17]

and Serre's Theorem on the homotopy groups of simply-connected finite CW complexes [21], we may assume that  $K'_0$  is a finite CW-complex. We then set

$$K_0 := \begin{cases} \bigvee_{i=1}^r S^n & n = 2, 4 \text{ or } n \text{ odd,} \\ K'_0 & \text{otherwise.} \end{cases}$$

**Lemma 4.8.** *Let  $G_1$  be a free abelian group of rank  $s$ , and let  $\bar{c}: \mathcal{G}_\epsilon^2 F \rightarrow G_1$  be a homomorphism. Then for  $i = 1, \dots, s$ , there are maps  $\phi_i: S^{2n-1} \rightarrow K_0$ , such for  $\phi := \sqcup_{i=1}^s \phi_i$ , the CW-complex*

$$K(\bar{c}) := K_0 \cup_\phi (\bigcup_{i=1}^s e^{2n})$$

*has  $(H^n(K; \mathbb{Z}), H^{2n}(K; \mathbb{Z}), \bar{c}_K) = (F, G_1, \bar{c})$ .*

*Proof.* We give a proof that applies for both definitions of  $K_0$ . We recall the  $i$ th- $\Gamma$ -group of a finite simply-connected CW-complex  $K$ , which is the group

$$\Gamma_i(K) := \text{Im}(\pi_i(K^{(i-1)}) \rightarrow \pi_i(K^{(i)})),$$

where  $K^{(i-1)} \rightarrow K^{(i)}$  is the inclusion of the  $(i-1)$ -skeleton of  $K$  into the  $i$ -skeleton of  $K$ . The  $\Gamma$  groups lie in Whitehead's long exact sequence

$$\cdots \rightarrow H_{i+1}(K; \mathbb{Z}) \xrightarrow{b} \Gamma_i(K) \xrightarrow{i_*} \pi_i(K) \xrightarrow{\rho} H_i(K; \mathbb{Z}) \rightarrow \cdots,$$

where  $i_*$  is the obvious inclusion,  $\rho$  is the Hurewicz homomorphism and  $b$  is a certain “boundary homomorphism”: see [3, Ch. 2]. Hence for  $K = K(F^\vee, n)$  we have

$$b: H_{2n}(K(F^\vee, n); \mathbb{Z}) \cong \Gamma_{2n-1}(K(F^\vee, n)),$$

and by [9, Theorem 3.4.3] there is a natural surjective homomorphism

$$Q: H_{2n}(K(F^\vee, n); \mathbb{Z}) \rightarrow (\mathcal{G}_\epsilon^2 F)^\vee,$$

which is given by taking the cup squares of elements in  $F = H^n(K(F^\vee, n); \mathbb{Z})$  and evaluating against  $H_{2n}(K(F^\vee, n); \mathbb{Z})$ .

Now consider  $i_{0*}: \pi_{2n-1}(K_0) \rightarrow \Gamma_{2n-1}(K(F^\vee, n))$ . We claim that  $Q \circ b^{-1} \circ i_{0*}$  is onto. When  $K_0 = K(F^\vee, n)^{(2n-2)}$ , we have that  $i_{0*}$  is onto by definition and so  $Q \circ b^{-1} \circ i_{0*}$  is onto. When  $K_0 = \bigvee_{i=1}^r S^n$ , we let  $\{x_1, \dots, x_r\}$  be a basis for  $F$ . For  $i \neq j$ , the element  $[x_i x_j] \in \mathcal{G}_\epsilon^2 F$  can be realised by Whitehead products  $[\iota_i, \iota_j]$  where  $\iota_k: S^n \rightarrow \bigvee_{i=1}^r S^n$  is the inclusion of the  $k$ th summand. When  $n = 2, 4$ , elements of the form  $[x_i^2] \in \mathcal{G}_\epsilon^2 F$  can be realised by maps  $\iota_i \circ h$ , where  $h: S^{2n-1} \rightarrow S^n$  has Hopf-invariant 1. For more details see [20, Proposition 3.11] in the case  $n = 2$ , the case  $n = 4$  is analogous. Hence we choose  $\phi_i \in \pi_{2n-1}(K_0)$  such that

$$Q \circ b^{-1} \circ i_{0*}(\phi_i) = \bar{c}^\vee(y_i^\vee),$$

where  $\bar{c}^\vee: G_1^\vee \rightarrow (\mathcal{G}_\epsilon^2 F)^\vee$  and  $\{y_1^\vee, \dots, y_s^\vee\}$  is a basis for  $G_1^\vee$ . By construction, we may then identify  $H^{2n}(K(\bar{c}); \mathbb{Z}) = G_1$  and the cup product structure on  $K(\bar{c})$  is given by  $\bar{c}_K = \bar{c}$ .  $\square$

When  $n = 2$ , the construction of the manifolds  $W = W(F, G_1, \lambda_1)$  follows easily from the results of Schmitt [20, §3] which build on handlebody theory and classical embedding results of Haefliger. When  $K_0 = K(F^\vee, n)^{(2n-2)}$ , we may have many layers of handles to attach, and it is convenient to use the theory of thickenings as developed by Wall [26]. We briefly recall the notion of a thickening: Let  $K$  be a simply-connected finite connected CW-complex. An  $m$ -thickening of  $K$  is a pair  $(W, \phi)$  where  $W$  is a compact  $m$ -manifold with simply-connected boundary  $\partial W$  and  $\phi: K \rightarrow W$  is a homotopy equivalence. Since the map  $\phi$  will be clear in our arguments from the discussion, we suppress it and from the notation and call  $W$  a thickening of  $K$ .

Now let  $(F, G_1, \bar{c}_1, \lambda_1)$  be as in the hypotheses of Lemma 4.6 and apply Lemma 4.8 to obtain the  $2n$ -complex  $K = K(F, G_1, \bar{c})$ . By [26, §3 Trivial thickening], there are unique  $4n$ -thickenings  $W_0(K)$  of  $K$  and  $W_0(K_0)$  of  $K_0$  which are compact submanifolds of  $\mathbb{R}^{4n}$  and which are called *trivial thickenings*. Moreover, by [26, Suspension Theorem],  $W_0(K) \cong W'_0(K) \times D^1$ , where  $W'_0(K) \subset \mathbb{R}^{4n-1}$  also thickens  $K$ . It follows that  $\lambda_{W_0(K)} = 0$  is the trivial form. By assumption, the required form  $\lambda_1$  on  $W(F, G_1, \bar{c}_1, \lambda_1)$  is an even form, so it suffices to show how to modify the intersection form of  $W_0(K)$  by any even form, without changing the cup-product structure.

By construction,  $W_0(K) = W_0(K_0) \cup_{\phi} (\cup_{i=1}^s h_i^{2n})$  is obtained from the trivial thickening of  $K_0$  by attaching  $s$   $2n$ -handles  $h_i^{2n} \cong D^{2n} \times D^{2n}$  along a framed embedding

$$\phi: \prod_{i=1}^s (D^{2n} \times S^{2n-1}) \hookrightarrow \partial W_0(K_0),$$

where we attach one handle for each element of a basis of  $G_1$ , which we assume has rank  $s$ . Now by [24, Lemma 1], every even symmetric bilinear form  $l$  is realised as the intersection form of handlebody  $W_l = D^{4n} \cup_{\phi_0} (\cup_{i=1}^s h_{0i}^{2n})$  which is obtained by attaching  $2n$ -handles  $h_{0i}^{2n}$  along a framed embedding

$$\phi_0: \prod_{i=1}^s (D^{2n} \times S^{2n-1}) \hookrightarrow D^{4n-1} \subset S^{4n-1}.$$

We take  $W_l$  to have the intersection form  $(G_1, \lambda_1)$ . Fixing an embedding  $D^{4n-1} \hookrightarrow \partial W_0(K_0)$  disjoint from  $\text{Im}(\phi)$ , we then form the framed embedding  $\phi' = \phi + \phi_0$  by tubing together the components of  $\phi$  and  $\phi_0$ . We define

$$W(F, G_1, \bar{c}_1, \lambda_1) := W_0(K_0) \cup_{\phi'} (\cup_{i=1}^s h_i^{2n})$$

to be the manifold obtained by attaching  $2n$ -handles to  $W_0(K_0)$  along  $\phi'$ . Since  $\text{Im}(\phi_0) \subset D^{4n-1}$ ,  $W = W(F, G_1, \bar{c}_1, \lambda_1)$  has that same homotopy type as  $W_0(K)$  and hence the same cup-product structure. On the other hand, the intersection form of  $W$  is identified with the intersection form of  $W_l$  which is the intersection form required for Lemma 4.6. This completes the proof of Lemma 4.6.

## 5. APPLICATIONS

In this section we discuss applications of the Bianchi-Massey tensor. We begin with the proof of Theorem 1.13 from the introduction, and then give examples where the Bianchi-Massey tensor is non-trivial despite all Massey products vanishing; these applications essentially reduce to understanding how the space  $\mathcal{B}_\epsilon[E]$  on which the Bianchi-Massey tensor is defined depends on the kernel  $E$  of the cup-square map. In the final section we briefly discuss the role of Bianchi-Massey tensor in the classification of simply-connected spin 7-manifolds.

**5.1. Intrinsic formality and the hard Lefschetz property.** We now prove Theorem 1.13, on the intrinsic formality of closed  $(n-1)$ -connected  $(4n-1)$ -manifolds with  $b_3 \leq 3$  and a hard Lefschetz property. In view of Corollary 1.12 it suffices to prove that  $\mathcal{B}_n(M) = 0$ . By Poincaré duality, the hard Lefschetz property is equivalent to equivalent to  $\text{Ann } E_n \subseteq \text{Grad}_\epsilon^2 H^n(M)^*$  containing a non-degenerate bilinear form  $q$ . Hence Theorem 1.13 is a consequence of the following algebraic result.

**Proposition 5.1.** *Let  $V$  be a vector space of dimension  $\leq 3$ , and  $E \subseteq \mathcal{G}_\epsilon^2 V$ . If  $\text{Ann } E \subseteq \text{Grad}_\epsilon^2 V^*$  contains a non-degenerate element  $q$ , then  $\mathcal{B}_\epsilon[E] = 0$ .*

*Proof.* It is convenient to consider the dual picture. By the duality of the sequences (7),  $\mathcal{B}_\epsilon[E] = 0$  if and only if the restriction of  $\phi: \text{Sym}^2 \text{Grad}_\epsilon^2 V^* \rightarrow \check{\mathcal{B}}_+(V^*)$  to  $\text{Ann } P^2 E \subseteq \text{Sym}^2 \text{Grad}_\epsilon^2 V^*$  is surjective. In terms of the Kulkarni-Nomizu product  $\odot$  described in Remark 2.6, the image of  $\phi$  is  $\text{Ann } E \odot \text{Grad}_\epsilon^2 V^*$ .

The case when  $n$  is odd is essentially trivial, because then  $q$  is a symplectic form on  $V$  and so  $\dim V = 0$  or  $2$ . If  $\dim V = 2$  then  $\check{\mathcal{B}}_+(V^*)$  is one-dimensional, and it is easy to see that  $q \odot q$  is non-zero. In particular  $q \odot \text{Alt}^2 V^* = \check{\mathcal{B}}_+(V^*)$ , so  $\mathcal{B}_-[E] = 0$ .

For the case when  $n$  is even, Besse [4, 1.119] explains that  $q \odot \text{Sym}^2 V^*$  is all of  $\check{\mathcal{B}}_-(V^*)$  for  $\dim V \leq 3$  (this is the same algebraic result that leads to the well-known fact from Riemannian geometry that the Riemann curvature is determined by the Ricci curvature in dimension  $\leq 3$ ).  $\square$

**5.2. Bianchi-Massey tensors without Massey products.** Let us now consider the question of when the Bianchi-Massey tensor can be non-trivial even though all Massey triple products vanish. According to (11), Massey triple products of degree  $n$  classes on a closed oriented  $(4n-1)$ -manifold  $M$  correspond to evaluating the Bianchi-Massey tensor on elements of  $\mathcal{B}_n(M) = \mathcal{B}_\epsilon[E_n]$  that are ordinary in the sense of (10). Therefore the algebraic version of the question is whether for a given vector space  $V := H^n(M)$  there exist subspaces  $E \subseteq \mathcal{G}_\epsilon^2 V$  such that  $\mathcal{B}_\epsilon[E]$  is not generated by ordinary elements.

Let us begin with the case when  $n$  is even. Example 5.3 gives an example where  $r := \dim V = 5$ , and  $\mathcal{B}_+[E]$  is non-trivial even though it contains *no* ordinary elements at all. We find it helpful to first present a dimension-counting argument to show that this is not an uncommon phenomenon when  $r > 5$ .

**Lemma 5.2.** *For any  $r := \dim V \geq 6$  there are  $E \subseteq P^2 V$  such that  $\mathcal{B}_+[E]$  is non-trivial but contains no ordinary elements.*

*Proof.* Note that the condition that  $\mathcal{B}_+[E]$  contain an ordinary element means that there are some 2-planes  $A, B \subseteq V$  such that  $AB := \{xy \mid x \in A, y \in B\} \subseteq P^2 V$  is contained in  $E$ . Let us first consider the case when  $A \neq B$ , so that  $AB$  has dimension 4 rather than 3.  $Gr_2(V) \times Gr_2(V)$  has dimension  $4r - 8$ , and the space of  $k$ -planes  $E \subseteq P^2 V$  containing a fixed  $AB$  has dimension  $(k-4) \left( \binom{r+1}{2} - k \right)$ . So the space of  $k$ -planes containing some  $AB$  with  $A \neq B$  has positive codimension in  $Gr_k(P^2 V)$  if

$$(k-4) \left( \binom{r+1}{2} - k \right) + 4r - 8 < k \left( \binom{r+1}{2} - k \right)$$

which reduces to

$$r - 2 < \binom{r+1}{2} - k.$$

Similarly, for the the space of  $k$ -planes containing some  $P^2 A$  to have positive codimension in  $Gr_k(P^2 V)$  reduces to

$$2r - 4 < 3 \left( \binom{r+1}{2} - k \right),$$

which is weaker than the above condition. Thus if  $k \leq \binom{r}{2} + 1$  and  $E \in Gr_k(P^2 V)$  is generic then  $\mathcal{B}_+[E]$  contains no ordinary elements. Now for  $k := \binom{r}{2} + 1$  and  $r \geq 6$

$$\dim P^2 E = \binom{\binom{r}{2} + 2}{2} \geq \binom{k+3}{4} = \dim P^4 V.$$

Hence in this case any  $E \in Gr_k(P^2 V)$  has  $\mathcal{B}_+[E]$  non-trivial.  $\square$

When  $k = \binom{r}{2} + 2$ , *i.e.* when  $E$  has codimension  $r - 2$ , the expected codimension of the space of  $k$ -planes  $E \subset P^2 V$  containing a fixed  $AB$  equals the dimension of  $Gr_2(V) \times Gr_2(V)$ , and we would expect each  $E$  to contain a finite number of  $AB$ . We can turn this round as follows.

For each  $(A, B) \in Gr_2(V) \times Gr_2(V)$  and  $q \in \text{Ann } E \subset P^2 V^*$ , the condition that  $AB \subseteq E$  implies that  $A$  and  $B$  are orthogonal with respect to the bilinear form  $q$ , which imposes 4 constraints on  $(A, B)$ . Now recall that mapping an oriented 2-plane with orthonormal basis  $x, y$  (w.r.t. some standard inner product) to  $\langle x + iy \rangle \in \mathbb{P}(V \otimes \mathbb{C})$  embeds  $\widetilde{Gr}_2(V) \hookrightarrow \mathbb{C}P^{r-1}$  with image the quadric  $Q := \{z : \sum z_i^2 = 0\}$ . That  $A$  and  $B$  are  $q$ -orthogonal translates to the condition that the image of  $(A, B)$  in  $\mathbb{C}P^{r-1} \times \mathbb{C}P^{r-1}$  lies in the subset of  $Q \times Q$  cut out by  $q(z, z) = q(z, \bar{z}) = 0$ . These correspond to sections of the line bundles  $\mathcal{O}(1, 1)$  and  $\mathcal{O}(1, -1)$  respectively (but only the first is holomorphic).

Writing  $c_1$  and  $c_2$  for the generators of the  $H^2$  of the two  $\mathbb{C}P^{r-1}$  factors, we see that the topological intersection number of  $Q \times Q$  with  $r - 2$  such subsets is the  $(c_1 c_2)^{r-1}$  coefficient of  $(2c_1)(2c_2)(c_1 + c_2)^{r-2}(c_1 - c_2)^{r-2}$ , which equals  $\pm 4 \binom{r-2}{\frac{r}{2}-1}$  when  $r$  is even, and vanishes when  $r$  is odd. This counts each ordered pair of unoriented 2-planes  $(A, B)$  with  $AB \subseteq E$  4 times, and possibly with some cancelling signs.

- For  $r = 3$  we expect that a generic codimension 1 subspace  $E \subset P^2V$  should contain no  $AB$ . Indeed, if the generator  $q \in \text{Ann } E$  is non-degenerate then there can obviously be no  $q$ -orthogonal 2-planes (and moreover we already argued in Proposition 5.1 that  $\mathcal{B}_+[E] = 0$  in that case).
- For  $r = 4$  we expect that for any  $E \subset P^2V$  of codimension 2, there should be at least two ordered pairs  $(A, B)$  of unoriented 2-planes such that  $AB$  is contained in  $E$  (and possibly only one unordered pair). If  $\text{Ann } E$  is spanned by two non-degenerate elements, then we could also see this by applying the Lefschetz fixed point theorem to the composition of the maps  $\perp: Gr_2(V) \rightarrow Gr_2(V)$  that they define (since  $\chi(Gr_2(V)) = 2$ ).  
If  $\text{Ann } E$  is spanned by  $x_1^2 + x_2^2 + x_3^2 + x_4^2$  and  $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2$  with  $\lambda_i$  distinct, then the coordinate planes corresponding to each partition of  $\{1, 2, 3, 4\}$  into two halves gives 6 ordered pairs of simultaneously orthogonal planes.
- For  $r = 5$  the calculation suggests that there may be some choices of  $q_1, q_2, q_3 \in \text{Sym}^2 V^*$  such that  $E := \text{Ann}\langle q_1, q_2, q_3 \rangle \subseteq P^2V$  does not contain any  $AB$ .

*Example 5.3.* Let  $E := \text{Ann}\langle q_1, q_2, q_3 \rangle \subseteq P^2\mathbb{Q}^5$  for  $q_i \in \text{Sym}^2\mathbb{Q}^5$  defined by

$$\begin{aligned} q_1 &= x_1x_4 + x_3x_5, \\ q_2 &= x_2x_5 + x_3x_4, \\ q_3 &= x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_4^2 + x_5^2. \end{aligned}$$

Suppose that  $A, B$  are orthogonal with respect to  $q_1$  and  $q_2$ . Let  $\pi := \{x_4 = x_5 = 0\} \subset \mathbb{Q}^5$ . One can check that

- (i) If  $A$  is contained in  $\pi$  then so is  $B$ , and vice versa.
- (ii) In fact, if  $A$  intersects  $\pi$  then  $B$  is contained in  $\pi$ , except if  $A \cap \pi$  is spanned by an element of the form  $(a^2, b^2, \pm ab, 0, 0)$ .
- (iii) If  $A$  and  $B$  are both transverse to  $\pi$  then they are equal.

Now if  $A$  and  $B$  are both contained in  $\pi$  then they are definitely not orthogonal with respect to  $q_3$ , because its restriction to  $\pi$  is non-degenerate. If  $A$  and  $B$  are equal they also cannot be  $q_3$ -orthogonal because  $q_3$  is positive-definite. Finally,

$$\begin{aligned} & q_3((a^2, b^2, \pm ab, 0, 0), (c^2, d^2, \pm cd, 0, 0)) \\ &= a^2c^2 + \frac{1}{2}(a^2d^2 + b^2c^2) + b^2d^2 \pm abcd \\ &= \frac{1}{2}(a^2 + b^2)(c^2 + d^2) + \frac{1}{2}(ac \pm bd)^2 > 0 \end{aligned}$$

for any non-zero  $(a, b)$  and  $(c, d)$ . Hence  $E$  contains no  $AB$ , but  $\mathcal{B}_+[E]$  has dimension at least  $\binom{13}{2} - \binom{8}{4} = 78 - 70 = 8$ .

We can also show that it happens for  $r \geq 11$  that  $\mathcal{B}_+[E]$  is non-trivial even though  $E$  does not even contain any monomials  $xy$ . The image of  $\mathbb{P}(V) \times \mathbb{P}(V) \rightarrow \mathbb{P}(P^2V)$ ,  $((x), (y)) \mapsto \langle xy \rangle$  has dimension  $2r - 2$ , so is disjoint from a generic  $E \subseteq P^2V$  of dimension  $k := \binom{r+1}{2} - 2r + 1$ . For  $r \geq 11$ , this makes  $\dim P^2E > \dim P^4V$ , so  $E$  automatically has  $\mathcal{B}_+[E]$  non-trivial.

Now let us consider the case when  $n$  is odd. In this case, the dimension count argument is not especially sharp. For  $r = 4$ , the fact that  $Gr_2(V)$  is 4-dimensional leads one to “expect” a generic 2-dimensional  $E \subseteq \Lambda^2V$  to contain (the one-dimensional)  $\Lambda^2A$  for some  $A \in Gr_2(V)$ . Nevertheless we have the following simple example.

*Example 5.4.* Let  $V = \mathbb{Q}^4$  and

$$E := \langle v_1 \wedge v_2 + v_3 \wedge v_4, v_1 \wedge v_3 - v_2 \wedge v_4, v_1 \wedge v_4 + v_2 \wedge v_3 \rangle \subset \Lambda^2V,$$

for a basis  $v_1, \dots, v_4 \in V$ . Then  $\mathcal{B}_-[E]$  does not contain any ordinary elements: indeed  $E$  contains no decomposable elements at all, so for any  $x, y \in V$ ,  $x \wedge y \in E$  implies that  $x$  and  $y$  are linearly dependent. However,  $\dim P^2E = 6$  while  $\dim \Lambda^4V = 1$ , so  $\dim \mathcal{B}_-[E] = 5$  (it is clear that  $P^2E$  maps onto  $\Lambda^4V$ ).

*Remark 5.5.* We emphasise that by Theorem 1.5, every triple  $(V, E, \mathcal{F} \in \mathcal{B}_\epsilon[E]^*)$  covered by Lemma 5.2 and Examples 5.3 and 5.4 is realised as  $(H^n(M), \ker(c_M), \mathcal{F}_M)$  for some closed  $(n-1)$ -connected  $(4n-1)$ -manifold  $M$ . A corresponding integral statement follows from Theorem 1.7.

**5.3. Some remarks on the classification of simply-connected spin 7-manifolds.** We begin this subsection by determining which linking models and Pontrjagin classes are realised by simply-connected spin 7-manifolds  $M$ . Let  $p_M \in H^4(M; \mathbb{Z})$  be the spin characteristic class, related to the first Pontrjagin class by  $2p_M = p_1(M)$ . By [7, Lemma 2.2],  $p_M$  is always even.

**Proposition 5.6.** *Given a linking model  $(F, G, b, \bar{c}, \bar{\mathcal{F}})$  and  $p \in 2G$ , there is a 1-connected spin 7-manifold  $M$  with*

$$(H^2(M; \mathbb{Z}), H^4(M; \mathbb{Z}), b_M, \bar{c}_M, \bar{\mathcal{F}}_M, p_M) = (F, G, b, \bar{c}, \bar{\mathcal{F}}, p)$$

*if and only if  $b$  and  $\bar{\mathcal{F}}$  are compatible. Moreover, we may always assume that  $TH^3(M; \mathbb{Z}) = 0$ .*

*Proof.* Given  $M$ , we first show that the linking form  $b_M$  and Bianchi-Massey tensor  $\mathcal{F}_M$  are compatible. Let  $K(H^2(M; \mathbb{Z}), 2)$  be the indicated Eilenberg-MacLane space. By [5, Proposition 4.2] the spin bordism group  $\Omega_7^{spin}(K(H^2(M; \mathbb{Z}), 2))$  vanishes. Hence  $M$  bounds over  $H^2(M; \mathbb{Z})$  and by Corollary 4.2  $b_M$  and  $\bar{\mathcal{F}}_M$  are compatible.

The existence statement for any linking model  $(F, G, b, \bar{c}, \bar{\mathcal{F}})$  with  $b$  and  $\bar{\mathcal{F}}$  compatible follows immediately from Theorem 4.4. Hence it remains to determine the possible values of  $p_M$ . By [20, §3] the manifolds  $M$  of Theorem 4.4 realising a given  $(F, G, \bar{c}, \bar{\mathcal{F}})$  can be assumed spin with  $p_M$  any element of  $2H^4(M; \mathbb{Z}) \cong 2G$ .  $\square$

Corollary 1.9 implies in particular that the invariants in Proposition 5.6 determine the diffeomorphism type of  $M$  up to a finite number of possibilities. We conclude with a discussion of how to pin down the remaining finite ambiguity.

The further invariants needed include the quadratic linking family and generalised Eells-Kuiper invariant from the 2-connected classification [7]. When  $M$  bounds over its normal 2-type in the sense of Kreck [17] (in particular, whenever  $\pi_2(M)$  is torsion-free), one can adapt the coboundary description of the Bianchi-Massey tensor from Lemma 4.1 to define mod  $m$  extensions of  $\bar{\mathcal{F}}$  for any integer  $m$ . One should further expect to be able to use such coboundaries to define some further generalised version of the generalised Kreck-Stolz invariants of Hepworth [15], which are based on [17, Theorem 6]. Assuming that  $\pi_2(M)$  is torsion-free, so that  $H^2(M; \mathbb{Z}/m) \cong H^2(M; \mathbb{Z}) \otimes \mathbb{Z}/m$ , may help evade some subtleties. (Note also that Proposition 5.6 lets us realise every possible integral cohomology ring compatible with  $\pi_2 M$  being torsion-free.)

**Conjecture 5.7** (cf. [8, §2.c]). *Simply-connected spin 7-manifolds  $M$  with torsion-free  $\pi_2(M)$  are classified up to spin diffeomorphism by the cohomology ring  $H^*(M; \mathbb{Z})$ , the spin characteristic class  $p_M$ , the torsion linking form  $b_M$  on  $TH^4(M; \mathbb{Z})$  and its family of quadratic refinements, the generalised Eells-Kuiper invariant from [7], the Bianchi-Massey tensor  $\bar{\mathcal{F}}_M$  and its mod  $m$  extensions, and some variation of the generalised Kreck-Stolz invariants from [15].*

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